Somewhat Homomorphic Encryption based on Random Ideal Codes

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Outline

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What is Homomorphic Encryption?

Public-key version

- $\mathsf{KeyGen}(1^\lambda) \to (\mathsf{pk}, \mathsf{sk})$
- $\mathsf{Enc}(m, \mathsf{pk}) \to \mathsf{ct}$
- $\mathsf{Dec}(\mathsf{ct}, \mathsf{sk}) \to m$
- $\mathsf{Eval}(f, \mathsf{ct}_1, \mathsf{ct}_2) \to \mathsf{ct}$

Proposition (Correctness)

$$\mathsf{Dec}(\mathsf{Eval}(f, \mathsf{Enc}(m_1, \mathsf{pk}), \mathsf{Enc}(m_2, \mathsf{pk})), \mathsf{sk}) = f(m_1, m_2)$$

What is Homomorphic Encryption?

- $f \in \{+, \times\} \to$ partial homomorphic encryption (RSA)
- $f \in \mathbb{F}_d[X] \to$ somewhat homomorphic encryption [BGN05]
- $f \in \mathbb{F}[X] \to$ fully homomorphic encryption [Gen09]

What is Homomorphic Encryption?

Secret-key version

- $\mathsf{KeyGen}(1^\lambda) \to \mathsf{sk}$
- $\mathsf{Enc}(m, \mathsf{sk}) \to \mathsf{ct}$
- $\mathsf{Dec}(\mathsf{ct}, \mathsf{sk}) \to m$
- $\mathsf{Eval}(f, \mathsf{ct}_1, \mathsf{ct}_2) \to \mathsf{ct}$

Proposition (Correctness)

$$\mathsf{Dec}(\mathsf{Eval}(f, \mathsf{Enc}(m_1, \mathsf{sk}), \mathsf{Enc}(m_2, \mathsf{sk})), \mathsf{sk}) = f(m_1, m_2)$$

Noisy ciphertexts

$$\mathrm{ct}_1 = \boldsymbol{m}_1 \boldsymbol{G} + \boldsymbol{e}_1$$
$$\mathrm{ct}_2 = \boldsymbol{m}_2 \boldsymbol{G} + \boldsymbol{e}_2$$

$$\mathrm{Eval}(+, \mathrm{ct}_1, \mathrm{ct}_2) = (\boldsymbol{m}_1 + \boldsymbol{m}_2)\boldsymbol{G} + \underbrace{\boldsymbol{e}_1 + \boldsymbol{e}_2}_{\text{weight} \approx 2w}$$

In general:

$$\mathrm{Eval}(f, \mathrm{Enc}(m_1, \mathrm{pk}), \mathrm{Enc}(m_2, \mathrm{pk})) \neq \mathrm{Enc}(f(m_1, m_2), \mathrm{pk}).$$

Bootstrapping: how to reduce ciphertext noise

$$(\mathsf{pk}_1, \mathsf{sk}_1) = \mathsf{KeyGen}(1^\lambda)$$
$$\mathsf{ct} = \mathsf{Enc}(m, \mathsf{pk}_1)$$

$$(\mathsf{pk}_2, \mathsf{sk}_2) = \mathsf{KeyGen}(1^\lambda)$$
$$\mathsf{ct}_1 = \mathsf{Enc}(\mathsf{Enc}(m, \mathsf{pk}_1), \mathsf{pk}_2)$$
$$\mathsf{ct}_2 = \mathsf{Enc}(\mathsf{sk}_1, \mathsf{pk}_2)$$

$$\mathsf{Eval}(\mathsf{Dec}(\cdot,\cdot), \mathsf{ct}_1, \mathsf{ct}_2) = \ \textbf{?}$$

Bootstrapping: how to reduce ciphertext noise

$$\mathsf{ct}_1 = \mathsf{Enc}(\mathsf{Enc}(m, \mathsf{pk}_1), \mathsf{pk}_2)$$
$$\mathsf{ct}_2 = \mathsf{Enc}(\mathsf{sk}_1, \mathsf{pk}_2)$$

$$\mathsf{Dec}(\mathsf{Eval}(\mathsf{Dec}(\cdot,\cdot), \mathsf{ct}_1, \mathsf{ct}_2), \mathsf{sk}_2) = \mathsf{Dec}(\mathsf{Enc}(m, \mathsf{pk}_1), \mathsf{sk}_1) = m$$

$$\big(\mathsf{Eval}(\mathsf{Dec}(\cdot,\cdot), \mathsf{ct}_1, \mathsf{ct}_2) \approx \mathsf{Enc}(m, \mathsf{pk}_2)\big)$$

History of fully homomorphic encryption

There has been a burst of activity in the last decade:

- 2009: Gentry's first FHE [Gen09]
- 2010-2015: Practical somewhat homomorphic encryption
- 2016: TFHE [CGGI16], bootstrapping below 100ms
- 2016-present: remarkable progress

... but most of existing constructions are based on **structured lattices**.

Outline

1. What is homomorphic encryption?
2. Existing homomorphic encryption with codes
3. New idea: homomorphic encryption with Alekhnovich framework
4. Our construction

Why homomorphic encryption with codes?

- An alternative to structured lattices
- Support and multi-dimensional approach
- Faster and simpler decryption circuit

Multi-dimensional approach for homomorphic encryption

$$\mathrm{ct}_1 = \boldsymbol{m}_1 \boldsymbol{G} \quad + \quad \boldsymbol{e}_1$$
$$\mathrm{ct}_2 = \boldsymbol{m}_2 \boldsymbol{G} \quad + \quad \underbrace{\boldsymbol{e}_2}_{\text{same support}}$$

$$\mathrm{Eval}(+, \mathrm{ct}_1, \mathrm{ct}_2) = (\boldsymbol{m}_1 + \boldsymbol{m}_2)\boldsymbol{G} + \underbrace{\boldsymbol{e}_1 + \boldsymbol{e}_2}_{\text{weight} \leq w}$$

Support Learning problem

Definition ([GHPT17])

Given a parity check matrix $\boldsymbol{H} \in \mathcal{M}_{n-k,n}(\mathbb{F}_q)$ and ℓ syndromes $\boldsymbol{s}_i = \boldsymbol{e}_i \boldsymbol{H}^T$ for $\boldsymbol{e}_i$ errors of weight w in the same support E, find E.

$\Longrightarrow$ restricts the number of independent ciphertexts than can be published.

Multiplication operation with codes

Technique from [AAPS11]:

$$\begin{aligned} \mathrm{ct}_1 &= \boldsymbol{m}_1 \boldsymbol{G} &+& \quad \boldsymbol{e}_1 \\ \mathrm{ct}_2 &= \boldsymbol{m}_2 \boldsymbol{G} &+& \quad \underbrace{\boldsymbol{e}_2}_{\text{same support}} \end{aligned}$$

$$\begin{aligned} \mathrm{Eval}(\times, \mathrm{ct}_1, \mathrm{ct}_2) &= \mathrm{ct}_1 \odot \mathrm{ct}_2 \\ &= \boldsymbol{m}_1 \boldsymbol{G} \odot \boldsymbol{m}_2 \boldsymbol{G} + \underbrace{\boldsymbol{e}_1 \odot \mathrm{ct}_2 + \boldsymbol{e}_2 \odot \mathrm{ct}_1 - \boldsymbol{e}_1 \odot \boldsymbol{e}_2}_{\text{still in the same support}} \end{aligned}$$

Evaluation codes

Definition

Let $\boldsymbol{g} = (g_1, \ldots, g_n)$ a vector of evaluation points, the evaluation code on $\boldsymbol{g}$ is

$$\mathcal{C} = \{(P(g_1), \ldots, P(g_n)) | P \in \mathcal{L}\}$$

Multiplication operation with evaluation codes

$$\begin{aligned} \mathsf{ct}_1 &= P_1(\boldsymbol{g}) + \boldsymbol{e}_1 \\ \mathsf{ct}_2 &= P_2(\boldsymbol{g}) + \underbrace{\boldsymbol{e}_2}_{\text{same support}} \end{aligned}$$

$$\begin{aligned} \mathsf{Eval}(\times, \mathsf{ct}_1, \mathsf{ct}_2) &= \mathsf{ct}_1 \odot \mathsf{ct}_2 \\ &= (P_1 \cdot P_2)(\boldsymbol{g}) + \underbrace{\boldsymbol{e}_1 \odot \mathsf{ct}_2 + \boldsymbol{e}_2 \odot \mathsf{ct}_1 - \boldsymbol{e}_1 \odot \boldsymbol{e}_2}_{\text{still in the same support}} \end{aligned}$$

Evaluation codes

Examples are:

- Reed-Muller [AAPS11]
- Reed-Solomon [BL11] (broken by [GOT12])

$\Longrightarrow$ highly **structured** codes

Outline

Alekhnovich (secret key version)

$$\mathrm{sk} = \boldsymbol{s}$$

$$\mathrm{Enc}(\boldsymbol{m}, \mathrm{sk}) = (\boldsymbol{G}, \boldsymbol{v} = \boldsymbol{s}\boldsymbol{G} + \boldsymbol{e} + \mathit{Encode}(\boldsymbol{m}))$$
$$\mathrm{Dec}(\mathrm{ct}, \mathrm{sk}) = \mathit{Decode}(\boldsymbol{v} - \boldsymbol{s}\boldsymbol{G})$$

Usually: $\mathit{Encode}(\boldsymbol{m}) = \boldsymbol{m}\mathcal{G}$, with $\mathcal{G}$ highly structured code

Ideal Alekhnovich (secret key version)

$$\text{sk} = \boldsymbol{s}$$

$$\text{Enc}(\boldsymbol{m}, \text{sk}) = (\boldsymbol{u}, \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{s} + \boldsymbol{e} + \mathit{Encode}(\boldsymbol{m}))$$
$$\text{Dec}(\text{ct}, \text{sk}) = \mathit{Decode}(\boldsymbol{v} - \boldsymbol{u} \cdot \boldsymbol{s})$$

Usually: $\mathit{Encode}(\boldsymbol{m}) = \boldsymbol{m}\mathcal{G}$, with $\mathcal{G}$ highly structured code

Security reduction (single ciphertext)

$$\mathrm{ct} = (\boldsymbol{u}, \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{s} + \boldsymbol{e} + Encode(\boldsymbol{m}))$$

$$\begin{pmatrix} \boldsymbol{v} \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{I}_n & \mathcal{IM}(\boldsymbol{u}) \end{array} \right) \begin{pmatrix} \boldsymbol{e} \\ \hline \boldsymbol{s} \end{pmatrix} + Encode(\boldsymbol{m})$$

Additive homomorphic properties

$$\text{ct}_1 = (\boldsymbol{u}_1, \boldsymbol{v}_1 = \boldsymbol{u}_1 \cdot \boldsymbol{s} + \boldsymbol{e}_1 + Encode(\boldsymbol{m}_1))$$

$$\text{ct}_2 = (\boldsymbol{u}_2, \boldsymbol{v}_2 = \boldsymbol{u}_2 \cdot \boldsymbol{s} + \boldsymbol{e}_2 + Encode(\boldsymbol{m}_2))$$

$$\text{ct}_+ = (\boldsymbol{u}_1 + \boldsymbol{u}_2, (\boldsymbol{u}_1 + \boldsymbol{u}_2) \cdot \boldsymbol{s} + \boldsymbol{e}_1 + \boldsymbol{e}_2 + Encode(\boldsymbol{m}_1 + \boldsymbol{m}_2))$$

Security reduction (two ciphertexts)

$$\mathrm{ct}_1 = (\boldsymbol{u}_1, \boldsymbol{v}_1 = \boldsymbol{u}_1 \cdot \boldsymbol{s} + \boldsymbol{e}_1 + Encode(\boldsymbol{m}_1))$$
$$\mathrm{ct}_2 = (\boldsymbol{u}_2, \boldsymbol{v}_2 = \boldsymbol{u}_2 \cdot \boldsymbol{s} + \boldsymbol{e}_2 + Encode(\boldsymbol{m}_2))$$

$$\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} = \left(\begin{array}{cc|c} \boldsymbol{I}_n & & \mathcal{IM}(\boldsymbol{u}_1) \\ & \boldsymbol{I}_n & \mathcal{IM}(\boldsymbol{u}_2) \end{array}\right) \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{s} \end{pmatrix} \begin{array}{c} + \; Encode(\boldsymbol{m}_1) \\ + \; Encode(\boldsymbol{m}_2) \end{array}$$

Security reduction (two ciphertexts)

$$\mathrm{ct}_1 = \boldsymbol{m}_1 \boldsymbol{G} + \boldsymbol{e}_1$$
$$\mathrm{ct}_2 = \boldsymbol{m}_2 \boldsymbol{G} + \boldsymbol{e}_2$$

$$\begin{pmatrix} \mathrm{ct}_1 & \mathrm{ct}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{G} \end{pmatrix}^{\begin{pmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 \end{pmatrix}} + \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix}$$

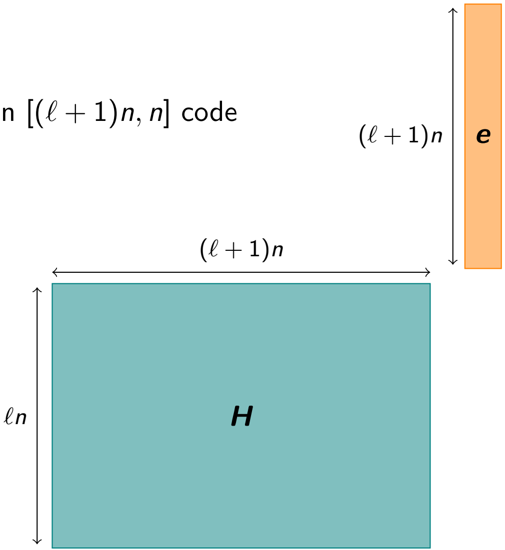
Security reduction (two ciphertexts)

$$\mathrm{ct}_1 = (\boldsymbol{u}_1, \boldsymbol{v}_1 = \boldsymbol{u}_1 \cdot \boldsymbol{s} + \boldsymbol{e}_1 + Encode(\boldsymbol{m}_1))$$
$$\mathrm{ct}_2 = (\boldsymbol{u}_2, \boldsymbol{v}_2 = \boldsymbol{u}_2 \cdot \boldsymbol{s} + \boldsymbol{e}_2 + Encode(\boldsymbol{m}_2))$$

$$\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} = \left(\begin{array}{cc|c} \boldsymbol{I}_n & & \mathcal{IM}(\boldsymbol{u}_1) \\ & \boldsymbol{I}_n & \mathcal{IM}(\boldsymbol{u}_2) \end{array}\right) \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{s} \end{pmatrix} \begin{array}{c} + \; Encode(\boldsymbol{m}_1) \\ + \; Encode(\boldsymbol{m}_2) \end{array}$$

Security reduction (ℓ ciphertexts)

Syndrome decoding problem in $[(\ell+1)n, n]$ code



Reminder: rank metric

	Hamming metric	Rank metric
Words	$(\mathbb{F}_q)^n$	$(\mathbb{F}_{q^m})^n$
Support	Indexes of non-zero coordinates	$\mathbb{F}_q$-subspace generated by coordinates
Small weight means...	Few non-zero coordinates	Each coordinate belongs to a small $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^m}$

Ideal rank-metric Alekhnovich (secret key version)

$$\text{sk} = \boldsymbol{s} \text{ (of small support } E\text{)}$$

$$\text{Enc}(\boldsymbol{m}, \text{sk}) = (\boldsymbol{u}, \boldsymbol{v} = \boldsymbol{u} \cdot \overbrace{\boldsymbol{s} + \boldsymbol{e}}^{\text{same support } E} + Encode(\boldsymbol{m}))$$

$$\text{Dec}(\text{ct}, \text{sk}) = Decode(\boldsymbol{v} - \boldsymbol{u} \cdot \boldsymbol{s})$$

Usually: $Encode(\boldsymbol{m}) = \boldsymbol{m}\mathcal{G}$, with $\mathcal{G}$ highly structured code

Outline

Encoding the message perpendicular to the error

$$\mathsf{sk} = \boldsymbol{s}$$

$$\mathsf{Enc}(\boldsymbol{m}, \mathsf{sk}) = (\boldsymbol{u}, \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{s} + \boldsymbol{e} + \mathit{Encode}(\boldsymbol{m}))$$
$$\mathsf{Dec}(\mathsf{ct}, \mathsf{sk}) = \mathit{Decode}(\boldsymbol{v} - \boldsymbol{u} \cdot \boldsymbol{s})$$

Usually: $\mathit{Encode}(\boldsymbol{m}) = \boldsymbol{m}\mathcal{G}$, with $\mathcal{G}$ highly structured code
This work: $\mathit{Encode}(\boldsymbol{m}) = \mathsf{e}^{\perp} \cdot \boldsymbol{m}$ with $\mathsf{e}^{\perp} \in E^{\perp}$

Encryption / decryption algorithms

Rank Somewhat Homomorphic Encryption (RankSHE)

$$\mathsf{sk} = \boldsymbol{s} \in \mathbb{F}_{q^m}^n, \mathrm{Supp}(\boldsymbol{s}) = E, e^{\perp} \in E^{\perp}, \langle e^{\perp}, e^{\perp} \rangle = 1$$

$$\mathrm{Enc}(\boldsymbol{m} \in \mathbb{F}_q^n, \mathsf{sk}) = (\boldsymbol{u} \in \mathbb{F}_{q^m}^n, \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{s} + \underbrace{\boldsymbol{e}}_{\boldsymbol{e} \in E} + e^{\perp} \cdot \boldsymbol{m})$$

$$\mathrm{Dec}((\boldsymbol{u}, \boldsymbol{v}), \mathsf{sk}) = \langle e^{\perp} \cdot \mathbf{1}, \boldsymbol{v} - \boldsymbol{u} \cdot \boldsymbol{s} \rangle$$

Proposition

The security of **RankSHE** *with ℓ independent ciphertexts is reduced to the $(\ell + 1)$-IRSD problem (decoding in an ideal $[(\ell + 1)n, n]_{q^m}$ code)*

Notation: scalar products

Let $(\gamma_1, \ldots, \gamma_m)$ be a $\mathbb{F}_q$-basis of $\mathbb{F}_{q^m}$.

Definition (Scalar product in $\mathbb{F}_{q^m}$)

For $x = \sum_i x^{(i)}\gamma_i \in \mathbb{F}_{q^m}$, $y = \sum_j y^{(j)}\gamma_j \in \mathbb{F}_{q^m}$

$$\left\langle \sum_i x^{(i)}\gamma_i, \sum_j y^{(j)}\gamma_j \right\rangle := \sum_i x^{(i)} y^{(i)} \in \mathbb{F}_q$$

Definition (Scalar product in $\mathbb{F}_{q^m}^n$)

For $\boldsymbol{x} = (x_i)_i \in \mathbb{F}_{q^m}^n, \boldsymbol{y} = (y_i)_i \in \mathbb{F}_{q^m}^n$,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := (\langle x_i, y_i \rangle)_i \in \mathbb{F}_q^n$$

Notation: scalar products

Lemma

For $u \in \mathbb{F}_{q^m}$, $\boldsymbol{v} \in \mathbb{F}_{q^m}^n$ *and* $\boldsymbol{m} \in \mathbb{F}_q^n$,

$$\langle u\mathbf{1}, \boldsymbol{m} \cdot \boldsymbol{v} \rangle = \boldsymbol{m} \cdot \langle u\mathbf{1}, \boldsymbol{v} \rangle.$$

Addition

$$\mathrm{Enc}(\boldsymbol{m}_1, \mathrm{sk}) + \mathrm{Enc}(\boldsymbol{m}_2, \mathrm{sk}) = \mathrm{Enc}(\boldsymbol{m}_1 + \boldsymbol{m}_2, \mathrm{sk})$$

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 \cdot \boldsymbol{s} + \boldsymbol{e}_1 + e^{\perp} \cdot \boldsymbol{m}_1$$

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 \cdot \boldsymbol{s} + \boldsymbol{e}_2 + e^{\perp} \cdot \boldsymbol{m}_2$$

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 = (\boldsymbol{u}_1 + \boldsymbol{u}_2) \cdot \boldsymbol{s} + \boldsymbol{e}_1 + \boldsymbol{e}_2 + e^{\perp} \cdot (\boldsymbol{m}_1 + \boldsymbol{m}_2)$$

Plaintext absorption

$$\boldsymbol{m}_1 \cdot \mathsf{Enc}(\boldsymbol{m}_2, \mathsf{sk}) = \mathsf{Enc}(\boldsymbol{m}_1 \cdot \boldsymbol{m}_2, \mathsf{sk})$$

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 \cdot \boldsymbol{s} + \boldsymbol{e}_2 + e^{\perp} \cdot \boldsymbol{m}_2$$

$$\boldsymbol{m}_1 \cdot \boldsymbol{v}_2 = (\boldsymbol{m}_1 \cdot \boldsymbol{u}_2) \cdot \boldsymbol{s} + \boldsymbol{m}_1 \cdot \boldsymbol{e}_2 + e^{\perp} \cdot (\boldsymbol{m}_1 \cdot \boldsymbol{m}_2)$$

Multiplication

$$\mathrm{Eval}(\times, (\boldsymbol{u}_1, \boldsymbol{v}_1), (\boldsymbol{u}_2, \boldsymbol{v}_2)) = (\boldsymbol{u}_1 \cdot \boldsymbol{u}_2, \boldsymbol{u}_1 \cdot \boldsymbol{v}_2 + \boldsymbol{u}_2 \cdot \boldsymbol{v}_1, \boldsymbol{v}_1 \cdot \boldsymbol{v}_2)$$

$$\begin{aligned}
&\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 \cdot \boldsymbol{s}^2 - (\boldsymbol{u}_1 \cdot \boldsymbol{v}_2 + \boldsymbol{u}_2 \cdot \boldsymbol{v}_1) \cdot \boldsymbol{s} + \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \\
&= (\boldsymbol{v}_1 - \boldsymbol{u}_1 \cdot \boldsymbol{s}) \cdot (\boldsymbol{v}_2 - \boldsymbol{u}_2 \cdot \boldsymbol{s}) \\
&= (\boldsymbol{e}_1 + e^{\perp} \cdot \boldsymbol{m}_1) \cdot (\boldsymbol{e}_2 + e^{\perp} \cdot \boldsymbol{m}_2) \\
&= \underbrace{\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 + e^{\perp} \cdot (\boldsymbol{e}_1 \cdot \boldsymbol{m}_2 + \boldsymbol{e}_2 \cdot \boldsymbol{m}_1)}_{\boldsymbol{e}',\ \mathrm{Supp}(\boldsymbol{e}') \subset E^2 \oplus e^{\perp} E} + (e^{\perp})^2 \cdot \boldsymbol{m}_1 \cdot \boldsymbol{m}_2
\end{aligned}$$

$$\mathrm{Dec}\mathit{AfterMul}((\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \mathrm{sk}) = \langle (e^{\perp})^2 \mathbf{1}, \boldsymbol{u} \cdot \boldsymbol{s}^2 - \boldsymbol{v} \cdot \boldsymbol{s} + \boldsymbol{w} \rangle$$

Summary

- Encryption scheme based on ideal random rank metric codes
- Unlimited additions
- Multiplication adds a component to the ciphertext and increases noise quadratically

Reducing ciphertext noise

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \text{ decryptable under } \mathsf{sk}_1$$

$$\Downarrow$$

$$(\boldsymbol{u}, \boldsymbol{v}) \text{ decryptable under } \mathsf{sk}_2$$

Bootstrapping

$$\mathsf{Eval}(\mathsf{Dec}\mathit{AfterMul}(\cdot,\cdot), \underbrace{\mathsf{ct}_1}_{\mathsf{Enc}((\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\mathsf{sk}_2)}, \underbrace{\mathsf{ct}_2}_{\mathsf{Enc}(\mathsf{sk}_1,\mathsf{sk}_2)}) \approx \mathsf{Enc}(\boldsymbol{m}, \mathsf{sk}_2)$$

$$\widehat{\mathsf{Eval}}(\mathsf{Dec}\mathit{AfterMul}(\cdot,\cdot), \underbrace{\mathsf{ct}_1}_{(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})}, \underbrace{\mathsf{ct}_2}_{\mathsf{Enc}(\phi(\mathsf{sk}_1),\mathsf{sk}_2)}) \approx \mathsf{Enc}(\boldsymbol{m}, \mathsf{sk}_2)$$

Bootstrapping

$$\text{Dec}\mathit{AfterMul}((\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \text{sk}_1) = \langle (e^{\perp})^2 \mathbf{1}, \boldsymbol{u} \cdot \boldsymbol{s}_1^2 - \boldsymbol{v} \cdot \boldsymbol{s}_1 + \boldsymbol{w} \rangle$$

$$\boldsymbol{u} = \sum_i \gamma_i \boldsymbol{u}^{(i)}$$

$$\boldsymbol{v} = \sum_i \gamma_i \boldsymbol{v}^{(i)}$$

$$\boldsymbol{w} = \sum_i \gamma_i \boldsymbol{w}^{(i)}$$

$$\text{Dec}\mathit{AfterMul}((\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \text{sk}_1) = \sum_i \underbrace{\boldsymbol{u}^{(i)}}_{\in \mathbb{F}_q} \cdot \underbrace{\boldsymbol{a}^{(i)}}_{\in \mathbb{F}_q} - \boldsymbol{v}^{(i)} \cdot \boldsymbol{b}^{(i)} + \boldsymbol{w}^{(i)} \cdot \boldsymbol{c}^{(i)}$$

Bootstrapping

$$\begin{aligned}\langle (e^{\perp})^2\mathbf{1}, \boldsymbol{u}\cdot \boldsymbol{s}_1^2\rangle &= \sum_i \langle (e^{\perp})^2\mathbf{1}, \gamma_i \boldsymbol{u}^{(i)} \cdot \boldsymbol{s}_1^2\rangle \\ &= \sum_i \langle (e^{\perp})^2\mathbf{1}, \boldsymbol{u}^{(i)} \cdot \gamma_i \boldsymbol{s}_1^2\rangle \\ &= \sum_i \boldsymbol{u}^{(i)} \cdot \langle (e^{\perp})^2\mathbf{1}, \gamma_i \boldsymbol{s}_1^2\rangle \\ &= \sum_i \boldsymbol{u}^{(i)} \cdot \boldsymbol{a}^{(i)}\end{aligned}$$

Bootstrapping

$$\boldsymbol{m} = \mathrm{Dec}\mathit{AfterMul}((\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \mathrm{sk}_1) = \sum_i \boldsymbol{u}^{(i)} \cdot \boldsymbol{a}^{(i)} - \boldsymbol{v}^{(i)} \cdot \boldsymbol{b}^{(i)} + \boldsymbol{w}^{(i)} \cdot \boldsymbol{c}^{(i)}$$

with

$$\boldsymbol{a}^{(i)} = \langle (e^{\perp})^2 \mathbf{1}, \gamma_i \boldsymbol{s}_1^2 \rangle$$
$$\boldsymbol{b}^{(i)} = \langle (e^{\perp})^2 \mathbf{1}, \gamma_i \boldsymbol{s}_1 \rangle$$
$$\boldsymbol{c}^{(i)} = \langle (e^{\perp})^2 \mathbf{1}, \gamma_i (1, 0, \ldots, 0) \rangle$$

Bootstrapping

$$\mathsf{ct}_{\boldsymbol{a}^{(i)}} = \mathsf{Enc}(\boldsymbol{a}^{(i)}, \mathsf{sk}_2) = \mathsf{Enc}(\langle (e^{\perp})^2 \mathbf{1}, \gamma_i \boldsymbol{s}_1^2 \rangle, \mathsf{sk}_2)$$

$$\mathsf{ct}_{\boldsymbol{b}^{(i)}} = \mathsf{Enc}(\boldsymbol{b}^{(i)}, \mathsf{sk}_2) = \mathsf{Enc}(\langle (e^{\perp})^2 \mathbf{1}, \gamma_i \boldsymbol{s}_1 \rangle, \mathsf{sk}_2)$$

$$\mathsf{ct}_{\boldsymbol{c}^{(i)}} = \mathsf{Enc}(\boldsymbol{c}^{(i)}, \mathsf{sk}_2) = \mathsf{Enc}(\langle (e^{\perp})^2 \mathbf{1}, \gamma_i (1, 0, \ldots, 0) \rangle, \mathsf{sk}_2)$$

$$\begin{aligned}
\widehat{\mathsf{Eval}}(\mathsf{Dec}\mathit{AfterMul}, (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \mathsf{ct}_{\boldsymbol{a}^{(i)}}, \mathsf{ct}_{\boldsymbol{b}^{(i)}}, \mathsf{ct}_{\boldsymbol{c}^{(i)}}) \\
:= \sum_i \boldsymbol{u}^{(i)} \cdot \mathsf{ct}_{\boldsymbol{a}^{(i)}} - \boldsymbol{v}^{(i)} \cdot \mathsf{ct}_{\boldsymbol{b}^{(i)}} + \boldsymbol{w}^{(i)} \cdot \mathsf{ct}_{\boldsymbol{c}^{(i)}} \\
= \mathsf{Enc}(\sum_i \boldsymbol{u}^{(i)} \cdot \boldsymbol{a}^{(i)} - \boldsymbol{v}^{(i)} \cdot \boldsymbol{b}^{(i)} + \boldsymbol{w}^{(i)} \cdot \boldsymbol{c}^{(i)}, \mathsf{sk}_2) \\
= \mathsf{Enc}(\boldsymbol{m}, \mathsf{sk}_2)
\end{aligned}$$

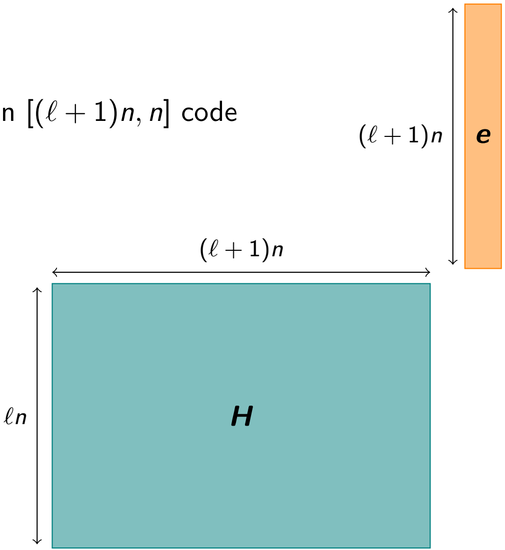
Bootstrapping

Our bootstrapping algorithm:

- Transforms a three-compenent ciphertext into a two-component ciphertext;
- reduces the noise from $\approx E_1^2$ to E_2;
- has no multiplicative cost;
- but... requires $3m$ independent ciphertexts under sk_2.

Security reduction (ℓ ciphertexts)

Syndrome decoding problem in $[(\ell+1)n, n]$ code



Security reduction (3m ciphertexts)

The attacker needs to solve the RSD problem in an ideal $[(3m+1)n, n]_{q^m}$ code.

There exists a polynomial attack [GRS13] in an $[n, k]_{q^m}$ code when

$$(k+1)(w+1) \leq n+1.$$

$\Longrightarrow$ maximal number of independent ciphertexts $\approx w$.

Reducing the number of bootstrapping ciphertexts

We pack several plaintexts into a single ciphertext:

$$\mathrm{Enc}((\boldsymbol{m}_1, \ldots, \boldsymbol{m}_p) \in (\mathbb{F}_q^n)^p, \mathrm{sk}) = (\boldsymbol{u} \in \mathbb{F}_{q^m}^n, \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{s} + \underbrace{\boldsymbol{e}}_{\|\boldsymbol{e}\| \leq w} + \sum_{i=1}^{p} \chi^i e^{\perp} \cdot \boldsymbol{m}_i)$$

with $\chi \in \mathbb{F}_{q^m}$ s.t. $\chi^p = 1$.

Maximal packing index $p = \frac{m}{w}$.

$\Longrightarrow$ reduces the number of bootstrapping plaintexts to $\frac{3m}{p} = 3w$.

Parameters

d	q	m	n	w	ℓ	Security	Key size	ct size	Add	Mul	Bootstrap
1	2	172	20	13	9	128	3.7 kB	**0.9 kB**	0.002 ms	**0.5 ms**	**2 ms**
2	2	367	183	7	5	128	17.0 kB	16.8 kB	0.04 ms	52 ms	374 ms
3	2	1296	314	6	4	128	210 kB	102 kB	0.3 ms	944 ms	11 s
4	2	3125	713	5	3	128	1.22 MB	557 kB	1 ms	14.3 s	239 s

Table: Example of paramaters for our SHE scheme, with associated sizes and execution timings. d is the number of possible multiplications. q, m and n are parameters of the rank linear code and w is the rank weight of the error. ℓ is the number of independant ciphertexts that can be published.

Comparison

Scheme	ct size	Bootstrap ct size	Mul time	Bootstrap time
TFHE [CGGI20]	2 kB	15.6 MB	0.03 ms	13 ms
[AAPS11]	18.5 kB	-	10 ms	-
This work	0.9 kB	35 kB	0.5 ms	2 ms

Table: Parameters are taken for 128-bit security, and for SHE schemes, with a single multiplication allowed.

Comparison

Scheme	ct size	Bootstrap ct size	Mul time	Bootstrap time
TFHE [CGGI20]	2 kB	15.6 MB	0.03 ms	13 ms
[AAPS11]	18.5 kB	-	10 ms	-
This work	0.9 kB	35 kB	0.5 ms	2 ms

Table: Parameters are taken for 128-bit security, and for SHE schemes, with a single multiplication allowed.

Thank you for your attention!

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