Decoding problems with hints and applications to multi-receiver code-based encryption schemes

Thomas Debris-Alazard[1], **Victor Dyseryn**[2], Duong Hieu Phan[2]

[1]Inria and LIX, Institut Polytechnique de Paris, École Polytechnique, Palaiseau, France
[2]Institut Polytechnique de Paris, Télécom Paris, Palaiseau, France

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Outline

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Decoding Problem

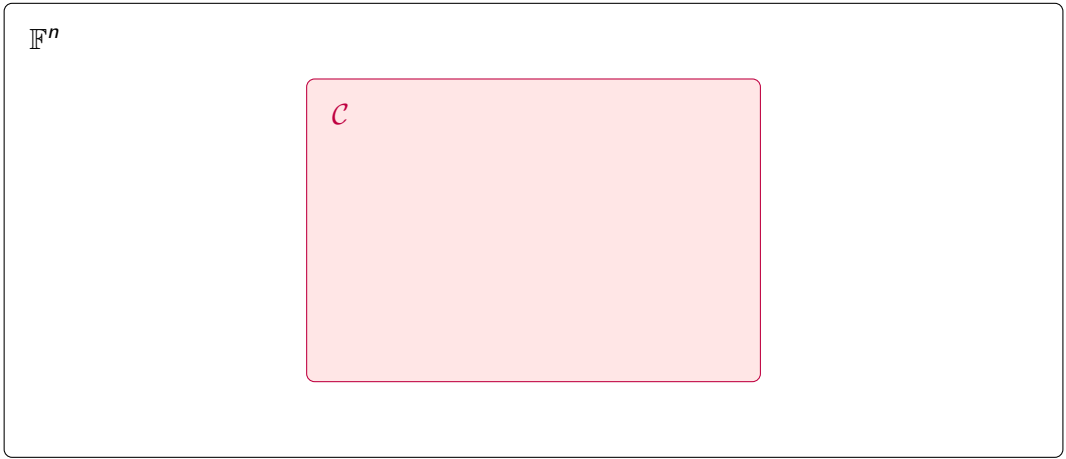
Distinguish between distributions

$$(\boldsymbol{G}, \boldsymbol{x}\boldsymbol{G} + \boldsymbol{e}) \quad \text{and} \quad (\boldsymbol{G}, \boldsymbol{u} + \boldsymbol{e})$$

for

- $\boldsymbol{G} \hookleftarrow \mathbb{F}^{k \times n}$
- $\boldsymbol{e} \hookleftarrow \mathcal{B}er(\omega)^n, \boldsymbol{x} \hookleftarrow \mathbb{F}^k$
- $\boldsymbol{u} \hookleftarrow \mathbb{F}^n$

Decoding Problem



Decoding Problem with hint

Distinguish between distributions

$$(\boldsymbol{v}, \boldsymbol{G}, \boldsymbol{x}\boldsymbol{G} + \boldsymbol{e}) \quad \text{and} \quad (\boldsymbol{v}, \boldsymbol{G}, \boldsymbol{u} + \boldsymbol{e})$$

for

- $\boldsymbol{v} \hookleftarrow \mathcal{B}er(\sigma)^n, \boldsymbol{G} \hookleftarrow \{\boldsymbol{M} \in \mathbb{F}^{k \times n} \mid \boldsymbol{M}\boldsymbol{v}^T = \boldsymbol{0}\}$
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Decoding Problem with hint

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- $\boldsymbol{e} \hookleftarrow \mathcal{B}er(\omega)^n, \boldsymbol{x} \hookleftarrow \mathbb{F}^k$
- $\boldsymbol{u} \hookleftarrow \langle \boldsymbol{v} \rangle^{\perp}$

Decoding Problem with ℓ hints

Distinguish between distributions

$$(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell, \boldsymbol{G}, \boldsymbol{x}\boldsymbol{G} + \boldsymbol{e}) \quad \text{and} \quad (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell, \boldsymbol{G}, \boldsymbol{u} + \boldsymbol{e})$$

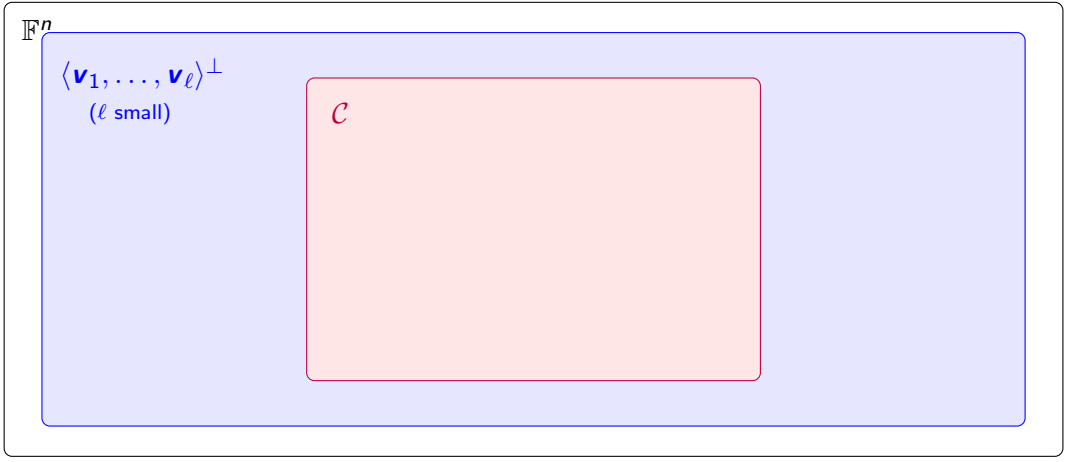
for

- $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \hookleftarrow \mathcal{B}er(\sigma)^n, \boldsymbol{G} \hookleftarrow \{\boldsymbol{M} \in \mathbb{F}^{k \times n} \mid \boldsymbol{M}\boldsymbol{v}_i^T = \boldsymbol{0} \text{ forall } i \in [1, \ell]\}$
- $\boldsymbol{e} \hookleftarrow \mathcal{B}er(\omega)^n, \boldsymbol{x} \hookleftarrow \mathbb{F}^k$
- $\boldsymbol{u} \hookleftarrow \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \rangle^\perp$

Analog to k-LWE problem [LPSS14]

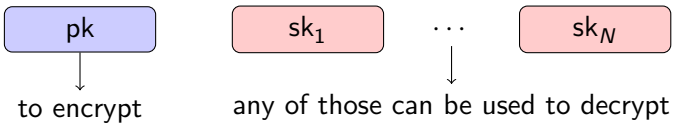
[LPSS14]: Ling, Phan, Stehlé, and Steinfeld. Hardness of k-LWE and applications in traitor tracing. (CRYPTO'14)

Decoding Problem with ℓ hints



Outline

What is a Multi-receiver PKE?

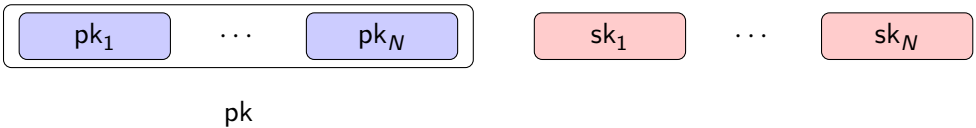


- $\mathsf{KeyGen}(1^\lambda, 1^N) \to (\mathsf{pk}, (\mathsf{sk}_1, \ldots, \mathsf{sk}_N))$

Properties:

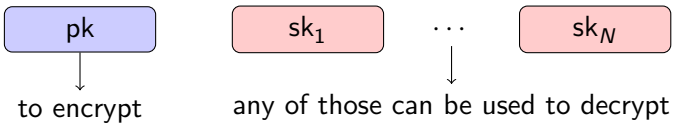
- Correctness: $\mathsf{Dec}\big(\mathsf{Enc}(m, \mathsf{pk}), \mathsf{sk}_i\big) = m$ for any $i \in [1, N]$
- IND-CPA security
- Compactness: $|\mathsf{ct}|, |\mathsf{pk}| \ll \lambda N$

Why compactness?



Not compact!

What is a Multi-receiver PKE?



- $\mathsf{KeyGen}(1^\lambda, 1^N) \to (\mathsf{pk}, (\mathsf{sk}_1, \ldots, \mathsf{sk}_N))$

Properties:

- Correctness: $\mathsf{Dec}\big(\mathsf{Enc}(m, \mathsf{pk}), \mathsf{sk}_i\big) = m$ for any $i \in [1, N]$
- IND-CPA security
- Compactness: $|\mathsf{ct}|, |\mathsf{pk}| \ll \lambda N$

A compact but trivial solution

- MultiPKE.KeyGen$(1^\lambda, 1^N)$:
  $(\mathsf{pk}, \mathsf{sk}) \leftarrow$ PKE.KeyGen(1^λ)
  Output $(\mathsf{pk}, (\mathsf{sk}, \ldots, \mathsf{sk}))$
- MultiPKE.Enc(m, pk) :
  Output $\mathsf{ct} =$ PKE.Enc(m, pk)
- MultiPKE.Dec$(\mathsf{ct}, \mathsf{sk})$:
  Output $m =$ PKE.Dec$(\mathsf{ct}, \mathsf{sk})$

We need more than bare IND-CPA security.
Force some difference between secret keys.

Multi-receiver PKE with revocation

- $\mathsf{MultiPKE.KeyGen}(1^\lambda, 1^N) \to (\mathsf{pk}, (\mathsf{sk}_1, \ldots, \mathsf{sk}_N))$
- $\mathsf{MultiPKE.Enc}\Big(m, \mathsf{pk}, \underbrace{(\mathsf{sk}_i)_{i\in\mathcal{C}}}_{\text{revoked users}}\Big) \to \mathsf{ct}$ (for a set $\mathcal{C} \subset [1, N]$)
- $\mathsf{MultiPKE.Dec}(\mathsf{ct}, \mathsf{sk}) \to m$

Properties:

- Correctness: $\mathsf{MultiPKE.Dec}\Big[\mathsf{MultiPKE.Enc}\Big(m, \mathsf{pk}, (\mathsf{sk}_i)_{i\in\mathcal{C}}\Big), \mathsf{sk}_j\Big] = m$ for any $j \in [1, N]\backslash\mathcal{C}$
- IND-CPA security with revocation
- Compactness: $|\mathsf{ct}|, |\mathsf{pk}| \ll \lambda N$

Alekhnovich PKE (1-bit scheme)

- PKE.KeyGen(1^λ) :

  $\boldsymbol{e} \hookleftarrow \mathcal{B}er(\sigma)^n$

  $\boldsymbol{G} \hookleftarrow \{\boldsymbol{M} \in \mathbb{F}^{k \times n} \mid \boldsymbol{M}\boldsymbol{e}^T = \boldsymbol{0}\}$

  Output $\mathsf{pk} = \boldsymbol{G}$ and $\mathsf{sk} = \boldsymbol{e}$

- PKE.Enc$(m \in \{0,1\}, \mathsf{pk} = \boldsymbol{G})$:

  $\boldsymbol{e}' \hookleftarrow \mathcal{B}er(\sigma)^n, \boldsymbol{x} \hookleftarrow \mathbb{F}^k$

  Output $\mathsf{ct} = \underset{(\text{if } m=0)}{\boldsymbol{x}\boldsymbol{G} + \boldsymbol{e}'}$ or $\underset{(\text{if } m=1)}{\boldsymbol{u} \hookleftarrow \mathbb{F}^n}$

- PKE.Dec$(\mathsf{ct}, \mathsf{sk} = \boldsymbol{e})$:

  Output $m = \mathsf{ct} \cdot \boldsymbol{e}^T$

Multi-receiver Alekhnovich PKE

- MultiPKE.KeyGen$(1^\lambda, 1^N)$:

  $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N \hookleftarrow \mathcal{B}er(\sigma)^n$
  $\boldsymbol{G} \hookleftarrow \{\boldsymbol{M} \in \mathbb{F}^{k \times n} \mid \boldsymbol{M}\boldsymbol{e}_i^T = \boldsymbol{0} \text{ forall } i\}$
  Output $\mathsf{pk} = \boldsymbol{G}$ and $(\mathsf{sk}_1, \ldots, \mathsf{sk}_N) = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N)$

- MultiPKE.Enc$(m \in \{0,1\}, \mathsf{pk} = \boldsymbol{G}, (\mathsf{sk}_i)_{i \in \mathcal{C}})$:

  $\boldsymbol{e}' \hookleftarrow \mathcal{B}er(\sigma)^n, \boldsymbol{x} \hookleftarrow \mathbb{F}^k, \boldsymbol{y} \hookleftarrow \langle (\mathsf{sk}_i)_{i \in \mathcal{C}} \rangle^\perp$
  Output $\mathsf{ct} = \underset{(\text{if } m = 0)}{\boldsymbol{x}\boldsymbol{G} + \boldsymbol{e}'}$ or $\underset{(\text{if } m = 1)}{\boldsymbol{y} + \boldsymbol{e}'}$

- PKE.Dec$(\mathsf{ct}, \mathsf{sk} = \boldsymbol{e})$:

  Output $m = \mathsf{ct} \cdot \boldsymbol{e}^T$

Inspired from lattice Dual-Regev-based multi-receiver scheme [LPSS14]

[LPSS14]: Ling, Phan, Stehlé, and Steinfeld. Hardness of k-LWE and applications in traitor tracing. (CRYPTO'14)

Decoding Problem with ℓ hints

Distinguish between distributions

$$(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell, \boldsymbol{G}, \boldsymbol{x}\boldsymbol{G} + \boldsymbol{e}) \quad \text{and} \quad (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell, \boldsymbol{G}, \boldsymbol{u} + \boldsymbol{e})$$

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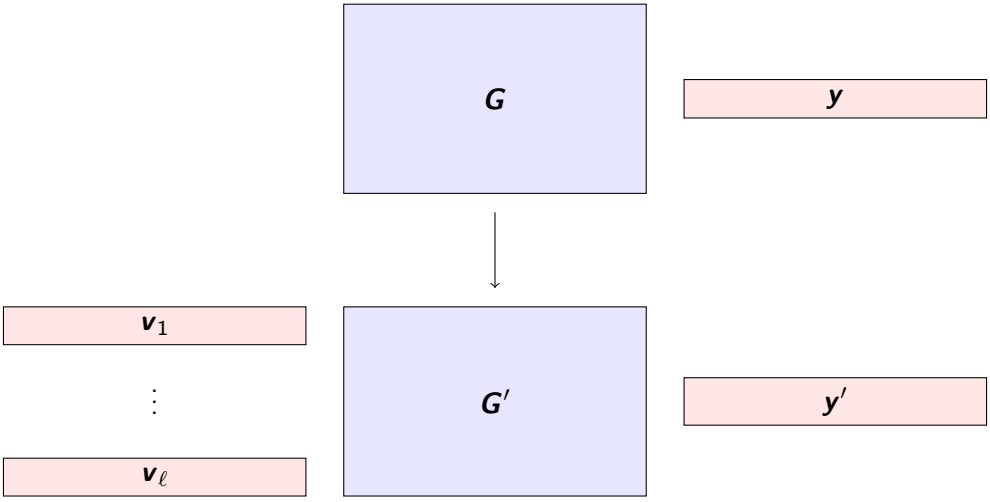
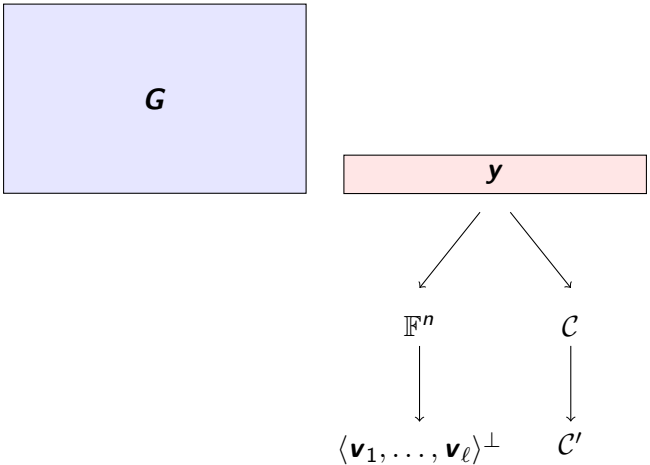
- $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \hookleftarrow \mathcal{B}er(\sigma)^n, \boldsymbol{G} \hookleftarrow \{\boldsymbol{M} \in \mathbb{F}^{k \times n} \mid \boldsymbol{M}\boldsymbol{v}_i^T = \boldsymbol{0} \text{ forall } i \in [1, \ell]\}$
- $\boldsymbol{e} \hookleftarrow \mathcal{B}er(\omega)^n, \boldsymbol{x} \hookleftarrow \mathbb{F}^k$
- $\boldsymbol{u} \hookleftarrow \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \rangle^\perp$

Outline

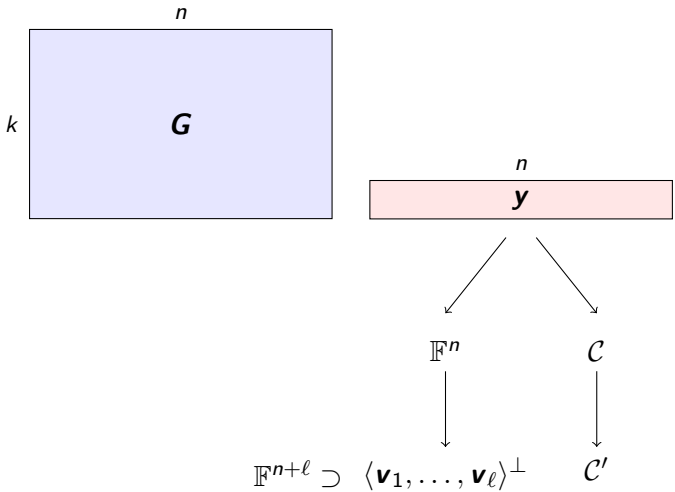
Reduction of security



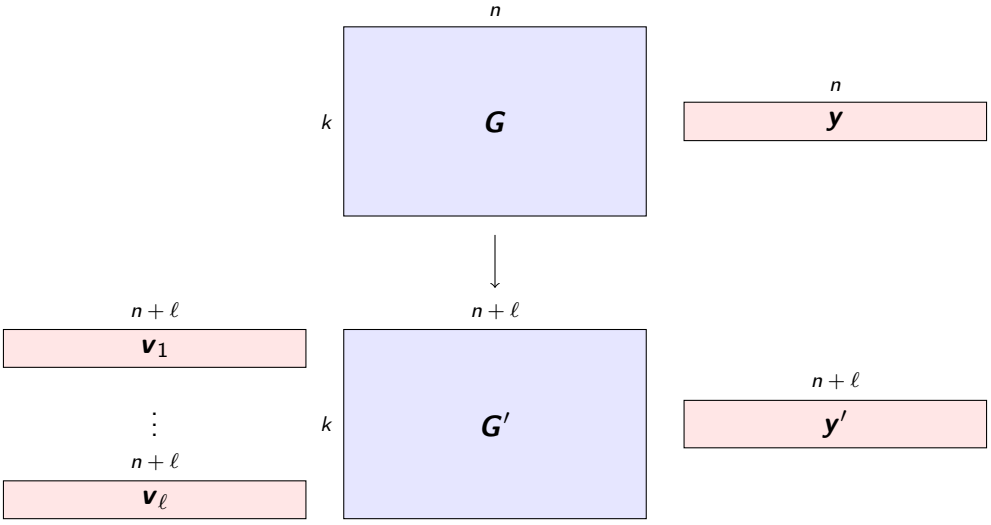
Reduction of security



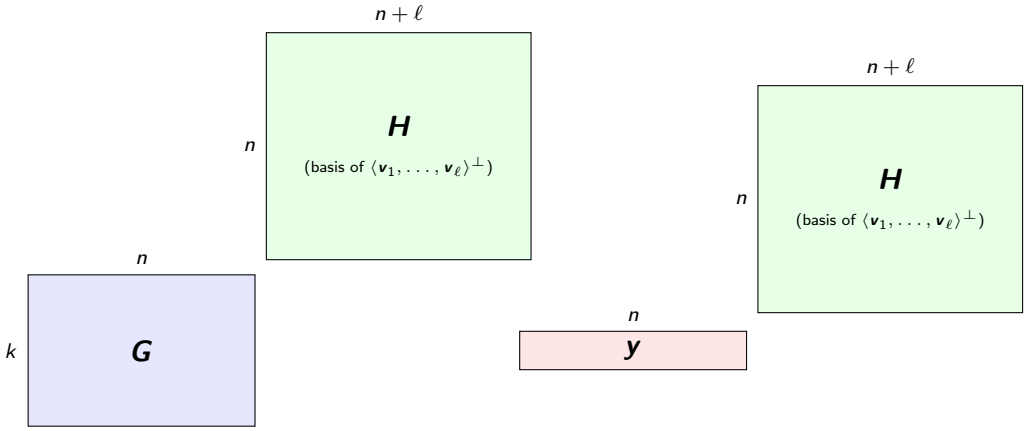
Reduction of security



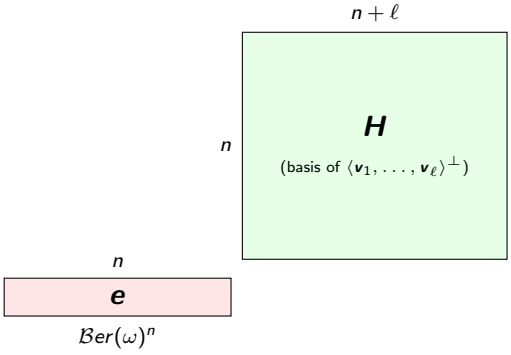
Reduction of security



Reduction of security



Uncorrelation of skewed Bernoulli noise



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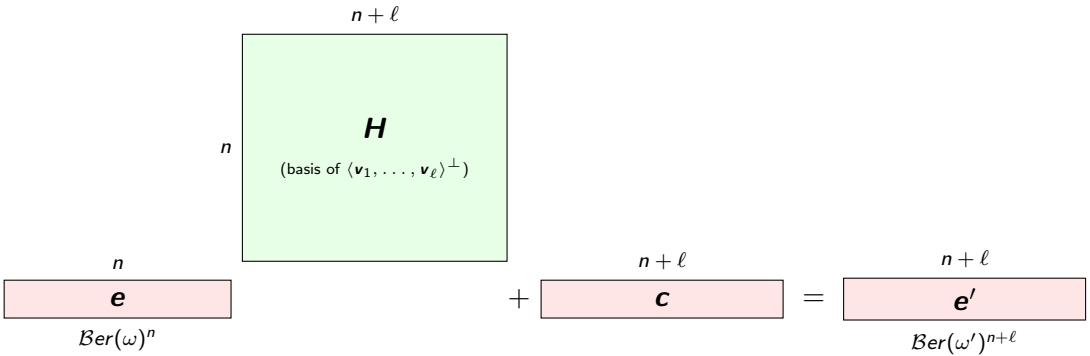

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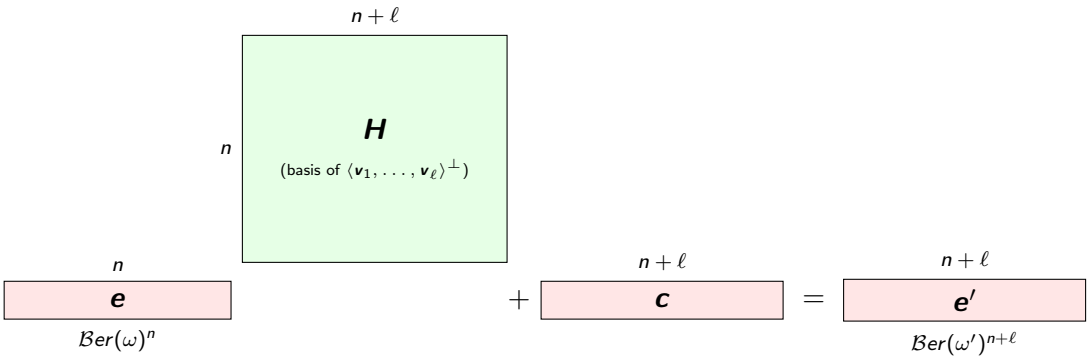

$$\omega' = 2\ell(2\omega)^{\frac{1}{2\ell}}$$

Theorem of the reduction of security

Theorem (Informal)

Consider noise levels ω *and* $\omega' \in \left(2\ell\,(2\omega)^{\frac{1}{2\ell}}, 1/2\right)$. *Then, the generic decoding problem* $\mathrm{DP}(n, k, \omega)$ *reduces to* ℓ-$\mathrm{DP}(n+\ell, k, \omega', \sigma)$.

Reduction is **non tight**; if $\ell \gg 1$, ω' tends quickly to $\frac{1}{2}$

Multi-receiver Alekhnovich PKE

- MultiPKE.KeyGen$(1^\lambda, 1^N)$:

  $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N \hookleftarrow \mathcal{B}er(\sigma)^n$
  $\boldsymbol{G} \hookleftarrow \{\boldsymbol{M} \in \mathbb{F}^{k \times n} \mid \boldsymbol{M}\boldsymbol{e}_i^T = \boldsymbol{0} \text{ forall } i\}$
  Output $\mathsf{pk} = \boldsymbol{G}$ and $(\mathsf{sk}_1, \ldots, \mathsf{sk}_N) = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N)$

- MultiPKE.Enc$(m \in \{0,1\}, \mathsf{pk} = \boldsymbol{G}, (\mathsf{sk}_i)_{i \in \mathcal{C}})$:

  $\boldsymbol{e}' \hookleftarrow \mathcal{B}er(\sigma)^n, \boldsymbol{x} \hookleftarrow \mathbb{F}^k, \boldsymbol{y} \hookleftarrow \langle (\mathsf{sk}_i)_{i \in \mathcal{C}} \rangle^\perp$
  Output $\mathsf{ct} = \underset{(\text{if } m = 0)}{\boldsymbol{x}\boldsymbol{G} + \boldsymbol{e}'}$ or $\underset{(\text{if } m = 1)}{\boldsymbol{y} + \boldsymbol{e}'}$

- PKE.Dec$(\mathsf{ct}, \mathsf{sk} = \boldsymbol{e})$:

  Output $m = \mathsf{ct} \cdot \boldsymbol{e}^T$

Perspectives: compact multi-receiver Alekhnovich PKE with MDPC codes

- MultiPKE.KeyGen$(1^\lambda, 1^N)$:

  $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N \hookleftarrow \mathcal{C}^{\perp}_{\text{MDPC}}$
  $\boldsymbol{G} \hookleftarrow \{\boldsymbol{M} \in \mathbb{F}^{k \times n} \mid \boldsymbol{M}\boldsymbol{e}_i^T = \boldsymbol{0} \text{ forall } i\}$
  Output $\text{pk} = \boldsymbol{G}$ and $(\text{sk}_1, \ldots, \text{sk}_N) = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N)$

- MultiPKE.Enc$(m \in \{0,1\}, \text{pk} = \boldsymbol{G}, (\text{sk}_i)_{i \in \mathcal{C}})$:

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- PKE.Dec$(\text{ct}, \text{sk} = \boldsymbol{e})$:

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MDPC-Decoding Problem with ℓ hints

Distinguish between distributions

$$(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell, \boldsymbol{G}, \boldsymbol{x}\boldsymbol{G} + \boldsymbol{e}) \quad \text{and} \quad (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell, \boldsymbol{G}, \boldsymbol{u} + \boldsymbol{e})$$

for

- $\boldsymbol{G} \hookleftarrow$ MDPC matrix, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \hookleftarrow \mathcal{C}_{\boldsymbol{G}}^{\perp}$ of short Hamming weight
- $\boldsymbol{e} \hookleftarrow \mathcal{B}er(\omega)^n, \boldsymbol{x} \hookleftarrow \mathbb{F}^k$
- $\boldsymbol{u} \hookleftarrow \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \rangle^{\perp}$

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- $\boldsymbol{e} \hookleftarrow \mathcal{B}er(\omega)^n, \boldsymbol{x} \hookleftarrow \mathbb{F}^k$
- $\boldsymbol{u} \hookleftarrow \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \rangle^{\perp}$

Thanks for your attention!

`https://eprint.iacr.org/2025/2151`

References I


San Ling, Duong Hieu Phan, Damien Stehlé, and Ron Steinfeld.
Hardness of k-LWE and applications in traitor tracing.
In Juan A. Garay and Rosario Gennaro, editors, CRYPTO 2014, Part I, volume 8616 of LNCS, pages 315–334. Springer, Heidelberg, August 2014.

IND-CPA security game with revocation

Adversary		**Challenger**
Chooses $\mathcal{C}$		$(\mathsf{pk}, (\mathsf{sk}_1, \ldots, \mathsf{sk}_N)) \leftarrow \mathsf{MultiPKE.KeyGen}(1^\lambda, 1^N)$
	$\mathcal{C} \longrightarrow$	
	$\longleftarrow \mathsf{pk}, (\mathsf{sk}_i)_{i \in \mathcal{C}}$	
Chooses m_0, m_1	$m_0, m_1 \longrightarrow$	
		Checks $\lvert m_0 \rvert = \lvert m_1 \rvert$
		$b \leftarrow\$ \{0, 1\}$
	$\longleftarrow \mathsf{MultiPKE.Enc}(m_b, \mathsf{pk}, (\mathsf{sk}_i)_{i \in \mathcal{C}})$	
Chooses b^*		

Adversary wins if $b = b^*$