Analysis of the security of the PSSI problem and cryptanalysis of Durandal signature scheme

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Families of post-quantum signatures

- Euclidean lattices
- Error-correcting codes
  - Hamming metric
  - Rank metric
- Isogenies
- Quadratic Multivariate
- Hash-based

Hamming metric

Definition (Hamming weight)

The Hamming weight of a word $\boldsymbol{x} \in (\mathbb{F}_q)^n$ is its number of non-zero coordinates :

$$w(\boldsymbol{x}) = \#\{i : x_i \neq 0\}$$

Definition (Hamming support)

The Hamming support of a word $\boldsymbol{x} \in (\mathbb{F}_q)^n$ is the set of indexes of its non-zero coordinates :

$$Supp(\boldsymbol{x}) = \{i : x_i \neq 0\}$$

Rank metric

In the rank metric, coordinates are in $\mathbb{F}_{q^m}$ (which is a field extension of $\mathbb{F}_q$ of degree m).

Definition (Rank weight)

Let $\gamma = (\gamma_1, ..., \gamma_m)$ be an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^m}$. A word $\boldsymbol{x} = (x_1, ..., x_n) \in (\mathbb{F}_{q^m})^n$ can be unfolded against γ :

$$\mathcal{M}(\boldsymbol{x}) = \begin{pmatrix} x_{1,1} & \dots & x_{n,1} \\ \vdots & & \vdots \\ x_{1,m} & \dots & x_{n,m} \end{pmatrix} \in \mathcal{M}_{m,n}(\mathbb{F}_q)$$

where $x_i = \sum_{j=1}^{m} x_{i,j}\gamma_j$.
The rank weight of x is defined as the rank of this matrix :

$$w_r(\boldsymbol{x}) = \text{rk } \mathcal{M}(\boldsymbol{x}) \in [0, \min(m, n)]$$

Rank metric

Definition (Rank support)

The support of a word $\boldsymbol{x} = (x_1, ..., x_n) \in (\mathbb{F}_{q^m})^n$ is the $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^m}$ generated by its coordinates :

$$Supp_r(\boldsymbol{x}) = Vect_{\mathbb{F}_q}(x_1, ..., x_n)$$

And likewise the Hamming metric, the rank weight is equal to the dimension of the rank support.

Difficult problems in code-based cryptography

Definition (Syndrome Decoding $\mathtt{SD}(n, k, w)$)

Given a random parity check matrix $\boldsymbol{H} \in \mathcal{M}_{n-k,n}(\mathbb{F}_q)$ and a syndrome $\boldsymbol{s} = \boldsymbol{He}$ for $\boldsymbol{e}$ an error of Hamming weight $w_h(\boldsymbol{e}) = w$, find $\boldsymbol{e}$.

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Durandal signature scheme

- Rank-based signature presented at EUROCRYPT'19 [ABG+19]
- Adaptation of Schnorr-Lyubashevsky proof of knowledge, with variations to avoid attacks
- Fiat-Shamir heuristic to transform into a signature scheme
- No equivalent found for Hamming metric
- Based on problems : RSL, IRSD, **PSSI**

Major types of post-quantum signatures

Hash and Sign

- Efficient
- Enables advanced protocols (IBE, ABE...)
- Hard to design

Fiat-Shamir

- Balanced performance
- Often based on ad-hoc difficult problems

Hash-based

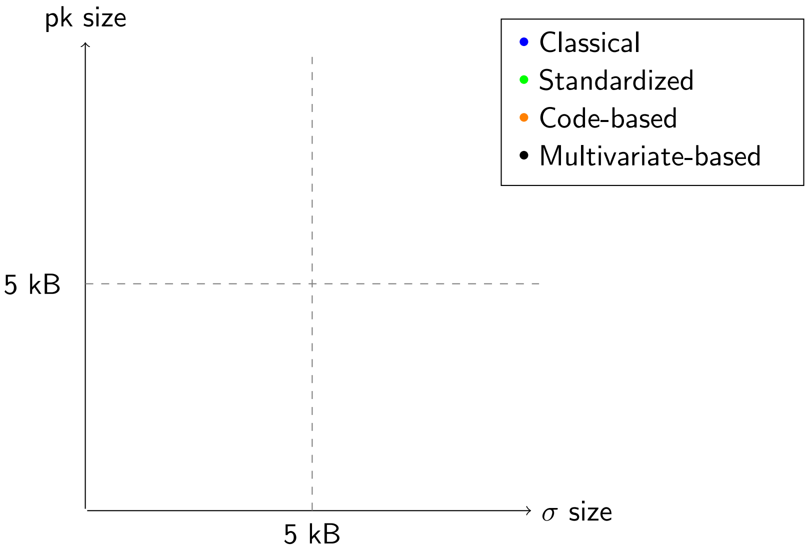
- High security
- Small public key
- Large signature size, slow to verify

Comparaison of post-quantum signatures

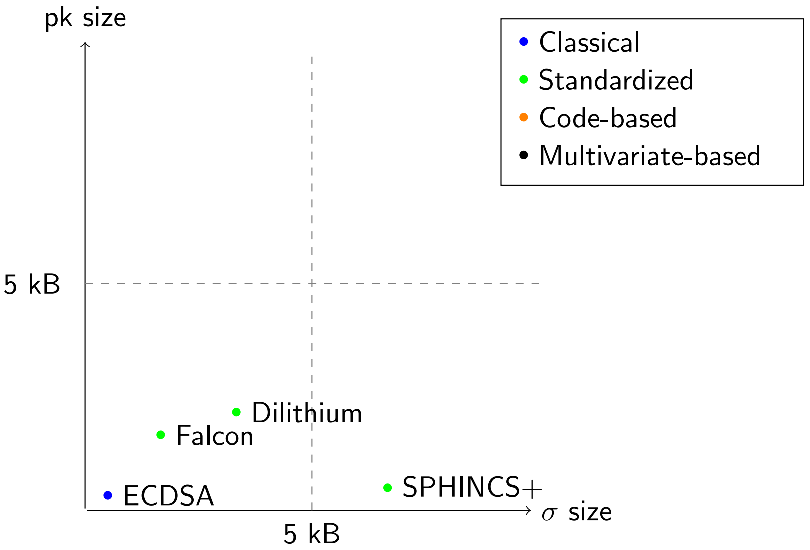
Name	Family	Type	pk size	σ size
ECDSA (Ed25519)	Classic		32B	64B
Falcon	Lattice	H&S	897B	666B
CRYSTALS-DILITHIUM	Lattice	F-S	1,3kB	2,4kB
WAVE [DAST19]	Hamming	H&S	3MB	1,6kB
SD-in-the-Head (3s) [FJR22]	Hamming	F-S	144B	8,5kB
Durandal-I	Rank	F-S	15.2kB	4.1kB
MAYO [Beu22]	Multivariate	H&S	518B	494B
SPHINCS+ (128s)	Hash		64B	8kB

Comparison of a few post-quantum signatures for 128 bits of security.

Comparaison of post-quantum signatures



Comparaison of post-quantum signatures



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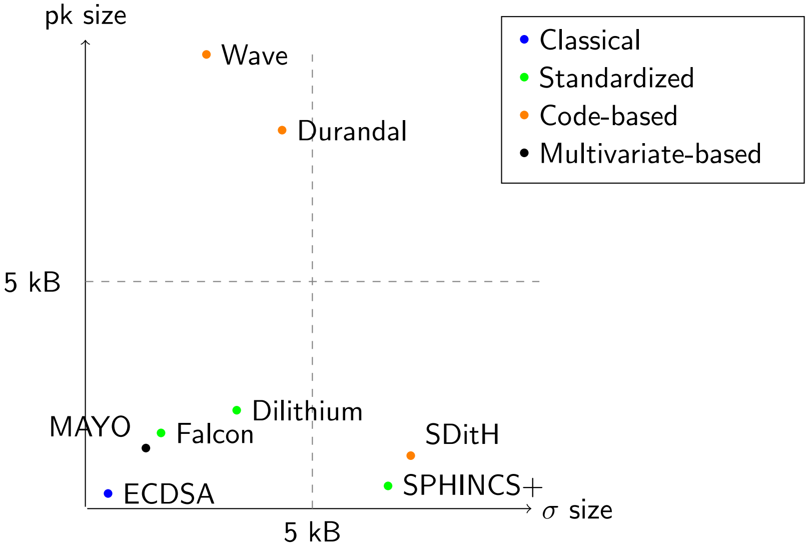

What has happened with Durandal since 2019 ?

- Resistant to attacks since 2019
- Better understanding of the RSL problem (algebraic attack in 2021 [BB21], combinatorial attack in 2022 [BBBG22])
- PSSI reduction to MinRank (ongoing work)
- New combinatorial attack on PSSI (this talk, breaks existing parameters in $\approx 2^{66}$ attempts)
- Optimizations and size-performance tradeoffs

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- Optimizations and size-performance tradeoffs

Summary

1. PSSI problem
2. A first observation
3. An attack against PSSI
4. Mitigation and new parameters
5. Conclusion and perspectives

Summary

1 PSSI problem

2 A first observation

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Notation

- $\mathbf{Gr}(d, \mathbb{F}_{q^m})$ is the set of subspaces of $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$-dimension d.
- $x \overset{\$}{\leftarrow} X$ means that x is chosen uniformly at random in X
- For E, F $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^m}$, the product space EF is defined as :

$$EF := Vect_{\mathbb{F}_q}\{ef | e \in E, f \in F\}$$

If $(e_1, ..., e_r)$ and $(f_1, ..., f_d)$ are basis of E and F, then $(e_i f_j)_{1 \leq i \leq r, 1 \leq j \leq d}$ contains a basis of EF.

Product space : example

Example

$(1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5)$ is a base of $\mathbb{F}_{2^6} \approx \mathbb{F}_2[\alpha]$.
As an exemple, let :

$$E = Vect\{1, \alpha\} = \{0, 1, \alpha, 1 + \alpha\}$$
$$F = Vect\{\alpha^2, \alpha^4\} = \{0, \alpha^2, \alpha^4, \alpha^2 + \alpha^4\}$$

$$EF = Vect\{\alpha^2, \alpha^3, \alpha^4, \alpha^5\}$$

PSSI problem

Definition (PSS sample)

Let $E \subset \mathbb{F}_{q^m}$ a subspace of $\mathbb{F}_q$-dimension r. A Product Space Subspace (PSS) sample is a pair of subspaces (F, Z) defined as follows :

- $F \stackrel{\$}{\leftarrow} \mathbf{Gr}(d, \mathbb{F}_{q^m})$
- $U \stackrel{\$}{\leftarrow} \mathbf{Gr}(rd - \lambda, EF)$ such that $\{ef | e \in E, f \in F\} \cap U = \{0\}$
- $W \stackrel{\$}{\leftarrow} \mathbf{Gr}(w, \mathbb{F}_{q^m})$
- $Z = W + U$

PSS sample : example

Example

We keep the same field $\mathbb{F}_{2^6} \approx \mathbb{F}_2[\alpha]$ with

$$E = Vect\{1, \alpha\} = \{0, 1, \alpha, 1 + \alpha\}$$
$$F = Vect\{\alpha^2, \alpha^4\} = \{0, \alpha^2, \alpha^4, \alpha^2 + \alpha^4\}$$

$$EF = Vect\{\alpha^2, \alpha^3, \alpha^4, \alpha^5\}$$

$$U = Vect\{\alpha^3 + \alpha^5\} \rightarrow \text{NOK}$$
$$U = Vect\{\alpha^2 + \alpha^5\} \rightarrow \text{OK}$$

PSSI problem

Definition (Random sample)

A random sample is a couple of subspaces (F, Z) with :

- $F \xleftarrow{\$} \mathbf{Gr}(d, \mathbb{F}_{q^m})$
- $Z \xleftarrow{\$} \mathbf{Gr}(w + rd - \lambda, \mathbb{F}_{q^m})$
- F and Z are independent

PSSI problem

Definition (PSSI problem, from Durandal [ABG+19])

The Product Spaces Subspaces Indistinguishability (PSSI) problem consists in deciding whether N samples (F_i, Z_i) are PSS samples or random samples.

Definition (Search-PSSI problem)

Given N PSS samples (F_i, Z_i), the search-PSSI problem consists in finding the vector space E of dimension r.

What happens if $\lambda = 0$?

There is no filtration : $(F, Z) = (F, W + EF)$.
Take $(f_1, ..., f_d)$ a basis of F.

To find E in one sample, compute :

$$A = \bigcap_{i=1}^{d} f_i^{-1} Z$$

Similar arguments than LRPC decoding :

$$\begin{aligned} f_i^{-1} Z &= f_i^{-1} f_1 E + ... + E + ... + f_i^{-1} f_d E + f_i^{-1} W \\ &= E + R_i \end{aligned}$$

Caveat : $\dim(Z)$ needs to be significantly lower than m.

Practical parameters for PSSI

m	w	r	d	λ
241	57	6	6	12

$$\text{Secret} : E \subset \mathbb{F}_{2^{241}}$$
$$\dim(E) = 6$$

$$\text{PSS sample} : (F, Z) \subset \mathbb{F}_{2^{241}}$$
$$\dim(F) = 6$$
$$\dim(Z) = 81$$
$$Z = W + U \text{ with } U \subsetneq EF$$

Existing attack for PSSI

Choose $A \subset F$ a subspace of dimension 2 and check whether

$$\dim(AZ) < 2(w + rd - \lambda)$$

Proposition ([ABG+19])

The advantage of the distinguisher is of the order of $q^{(rd-\lambda)-m}$.

Existing attack for PSSI

Choose $A \subset F$ a subspace of dimension 2 and check whether

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Proposition ([ABG⁺19])

The advantage of the distinguisher is of the order of $q^{(rd-\lambda)-m}$.

Several problems :

- The distinguisher only uses **one** signature ;
- It does not depend on w ;
- It does not allow to recover the secret space E.

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Combining two instances

Simplifying assumption : $w = 0$, m very large.

Combine two PSSI instances $(F_1, Z_1), (F_2, Z_2)$ by computing

$$A := F_1 Z_2 + F_2 Z_1 \subset E(F_1 F_2)$$

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$$A := F_1 Z_2 + F_2 Z_1 \subset E(F_1 F_2)$$

With great probability,

$$A = E(F_1 F_2)$$

($F_1 Z_2 + F_2 Z_1$ is **not** filtered in $E(F_1 F_2)$)

A partial explanation

If there exists $(e_1, e_2, f_1, f_1', f_2, f_2')$ such that

$$e_1 f_1 + e_2 f_1' = z_1 \in Z_1$$
$$e_1 f_2 + e_2 f_2' = z_2 \in Z_2$$

then

$$f_1' z_2 - f_2' z_1 = e_1 (f_1' f_2 - f_2' f_1)$$

Protection by m

Recall that

- $\dim F = d$
- $\dim Z = w + rd - \lambda$

so

$$\dim F_1 Z_2 + F_2 Z_1 = 2d(w + rd - \lambda) > m$$

but we can take subspaces of F_1 and F_2 to remain below m!

m	w	r	d	λ	$w + rd - \lambda$
241	57	6	6	12	81

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Refining the first observation

By drawing randomly

$$(f_1, f_1') \xleftarrow{\$} F_1, (f_2, f_2') \xleftarrow{\$} F_2$$

we get a possibility of having a product element ef (with $e \in E, f \in F_1F_2$) :

$$ef \in f_1{}' Z_2 + f_2' Z_1$$

We need :

- A way to recover this element $e \in E$;
- A precise probability of recovering e

Simultaneous 2-sums

If the attacker is lucky, after drawing random couples

$$(f_1, f_1') \xleftarrow{\$} F_1, (f_2, f_2') \xleftarrow{\$} F_2, (f_3, f_3') \xleftarrow{\$} F_3, (f_4, f_4') \xleftarrow{\$} F_4,$$

there exists a couple $(e, e') \in E^2$, such that a system $(\mathcal{S})$ of four conditions is verified :

$$(\mathcal{S}) : \begin{cases} ef_1 + e'f_1' = z_1 \in Z_1 \\ ef_2 + e'f_2' = z_2 \in Z_2 \\ ef_3 + e'f_3' = z_3 \in Z_3 \\ ef_4 + e'f_4' = z_4 \in Z_4 \end{cases}$$

Cramer formulas

$$(\mathcal{S}) : \begin{cases} ef_1 + e'f_1' = z_1 \in Z_1 \\ ef_2 + e'f_2' = z_2 \in Z_2 \\ ef_3 + e'f_3' = z_3 \in Z_3 \\ ef_4 + e'f_4' = z_4 \in Z_4 \end{cases}$$

$$e = \frac{\begin{vmatrix} z_i & f_i' \\ z_j & f_j' \end{vmatrix}}{\begin{vmatrix} f_i & f_i' \\ f_j & f_j' \end{vmatrix}}.$$

Cramer formulas

$$(\mathcal{S}) : \begin{cases} e f_1 + e' f_1' = z_1 \in Z_1 \\ e f_2 + e' f_2' = z_2 \in Z_2 \\ e f_3 + e' f_3' = z_3 \in Z_3 \\ e f_4 + e' f_4' = z_4 \in Z_4 \end{cases}$$

$$e \in A_{i,j} = \frac{\begin{vmatrix} Z_i & f_i' \\ Z_j & f_j' \end{vmatrix}}{\begin{vmatrix} f_i & f_i' \\ f_j & f_j' \end{vmatrix}} = \frac{f_j' Z_i + f_i' Z_j}{\begin{vmatrix} f_i & f_i' \\ f_j & f_j' \end{vmatrix}}.$$

Cramer formulas

$$(\mathcal{S}) : \begin{cases} e f_1 + e' f_1' = z_1 \in Z_1 \\ e f_2 + e' f_2' = z_2 \in Z_2 \\ e f_3 + e' f_3' = z_3 \in Z_3 \\ e f_4 + e' f_4' = z_4 \in Z_4 \end{cases}$$

$$\langle e \rangle = \bigcap_{i \neq j} \frac{\begin{vmatrix} Z_i & f_i' \\ Z_j & f_j' \end{vmatrix}}{\begin{vmatrix} f_i & f_i' \\ f_j & f_j' \end{vmatrix}}.$$

The attack

Input : Four PSSI samples $(F_1, Z_1), (F_2, Z_2), (F_3, Z_3), (F_4, Z_4)$

- Step 1 : Draw
  $(f_1, f_1') \xleftarrow{\$} F_1, (f_2, f_2') \xleftarrow{\$} F_2, (f_3, f_3') \xleftarrow{\$} F_3, (f_4, f_4') \xleftarrow{\$} F_4$
- Step 2 : Compute

$$B = \bigcap_{i \neq j} \frac{\begin{vmatrix} Z_i & f_i' \\ Z_j & f_j' \end{vmatrix}}{\begin{vmatrix} f_i & f_i' \\ f_j & f_j' \end{vmatrix}}.$$

- Step 3 : If $\dim(B) = 0$ or $\dim(B) > 1$, go back to Step 1.
- Step 4 : If $B = \langle e \rangle$, add e to E_{guess} and restart with new samples.

Probability of existence of 2-sums

Heuristic

Let $(e_1, e_2) \in E$ and $U \subset EF$ filtered of dimension $rd - \lambda$.
For $(f_1, f_2) \xleftarrow{\$} F$ the event

$$e_1 f_1 + e_2 f_2 \in U$$

happens with probability $q^{-\lambda}$.

Probability of existence of 2-sums

Lemma

Let $(f_i, f_i') \xleftarrow{\$} F_i$ *for* $i \in [1, 4]$. *Under the previous heuristic, and if* $\lambda = 2r$, *the probability* ε *that there exists a pair* $(e, e') \in E^2$, *such that the system* $(\mathcal{S})$ *of four conditions is verified*

$$(\mathcal{S}) : \begin{cases} ef_1 + e'f_1' = z_1 \in Z_1 \\ ef_2 + e'f_2' = z_2 \in Z_2 \\ ef_3 + e'f_3' = z_3 \in Z_3 \\ ef_4 + e'f_4' = z_4 \in Z_4 \end{cases}$$

admits an asymptotic development

$$\varepsilon = q^{-6r} + o_{r\to\infty}(q^{-10r})$$

Does this really work ?

We want the chain of intersections

$$B = \bigcap_{i \neq j} \frac{\begin{vmatrix} Z_i & f_i' \\ Z_j & f_j' \end{vmatrix}}{\begin{vmatrix} f_i & f_i' \\ f_j & f_j' \end{vmatrix}}.$$

to be equal to $\{0\}$, in general.

All the subspaces $f_i Z_j + f_j Z_i$ are of dimension $2(w + rd - \lambda)$.

m	w	r	d	λ	$2(w + rd - \lambda)$
241	57	6	6	12	162

Probabilities on the intersection of two vector spaces

Heuristic

Let A and B be uniformly random and independent subspaces of $\mathbb{F}_{q^m}$ of dimension a and b, respectively.

- If $a+b<m$, then $\mathbb{P}(\dim(A\cap B)>0)\approx q^{a+b-m}$;
- If $a+b\geq m$, then the most probable outcome is $\dim(A\cap B)=a+b-m$.

Generalization to n intersections

Heuristic

For $1 \leq i \leq n$, let $A_i \xleftarrow{\$} \mathbf{Gr}(a, \mathbb{F}_{q^m})$ be independent subspaces of fixed dimension a.

- If $na < (n-1)m$, then $\mathbb{P}(\dim(\bigcap_{i=1}^n A_i) > 0) \approx q^{na-(n-1)m}$;
- If $na \geq (n-1)m$, then the most probable outcome is $\dim(\bigcap_{i=1}^n A_i) = na - (n-1)m$;

In our setting :

- $a = 162, m = 241, n = 4$

$$\mathbb{P}(\dim(B) > 0) \approx q^{-75}$$

Total complexity of the attack

Proposition

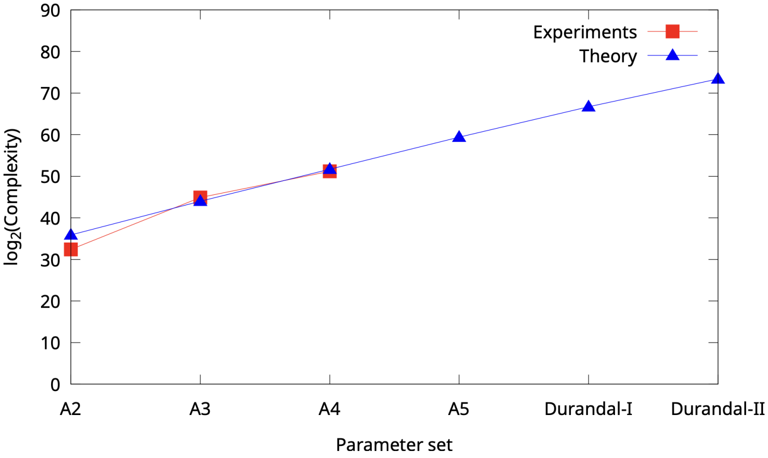
The average complexity of the attack is :

$$(r + \frac{1}{q-1}) \times 160m(w + rd - \lambda)^2 \times q^{6r}$$

operations in $\mathbb{F}_q$.

	Theoretical complexity
Durandal-I	2^{66}
Durandal-II	2^{73}

Experimental results



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Combinatorial factor of the attack

$$\approx q^{6r}$$

(when $\lambda = 2r$)

Increase λ	$\Rightarrow$	Impossible due to inexistence of solution
Decrease m	$\Rightarrow$	Impossible due to Singleton bound
Increase r	$\Rightarrow$	Very large parameters... ($m \geq 400$)

Increase q !

New parameters

q	m	k	n	w	r	d	λ
2	241	101	202	57	6	6	12

pk size	σ size	MaxMinors [BBC+20]	Our attack
15.2KB	4.1KB	98	56

$\downarrow$

q	m	k	n	w	r	d	λ
4	173	85	170	5	8	9	18

pk size	σ size	MaxMinors [BBC+20]	Our attack
14.7KB	5.1KB	232	128

Keygen	Signature	Verification
5ms	350ms	2ms

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Conclusion

- Analysis of a less studied problem at the core of a competitive signature scheme
- New secure parameters remain attractive
- Optimizations makes the scheme even more competitive

Perspectives

- Refine the analysis on the security of PSSI problem
- Tweak to avoid the new attack on PSSI without penalizing the parameters

Thank you for your attention !

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A partial explanation

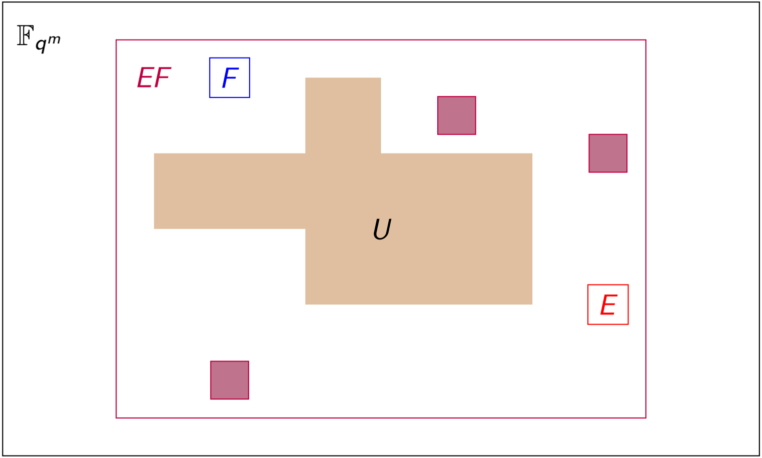
If there exists $(e_1, e_2) \in E^2$ such that

$$e_1 f_1 + e_2 f_1' = z_1 \in Z_1$$
$$e_1 f_2 + e_2 f_2' = z_2 \in Z_2$$

then

$$f_1' z_2 + f_2' z_1 = e_1(f_1' f_2 + f_2' f_1)$$

Impossibility to avoid 2-sums



Refining the first observation

By drawing randomly a matrix

$$\begin{pmatrix} f_1 & f_1' \\ f_2 & f_2' \end{pmatrix} \quad (f_1, f_1') \xleftarrow{\$} F_1, (f_2, f_2') \xleftarrow{\$} F_2$$

we get (roughly) q^{-4d} chances of having a product element ef (with $e \in E, f \in F_1 F_2$) :

$$ef \in f_1' Z_2 + f_2' Z_1$$

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We need :

- A way to recover this element $e \in E$;
- A precise probability of recovering e

The attack

We consider three samples :

$$(F_1, Z_1)$$
$$(F_2, Z_2)$$
$$(F_3, Z_3)$$

Let $(f_1, f_1') \xleftarrow{\$} F_1$. With probability greater than

$$(1 - 1/e)^3 \approx 0,25$$

there exists elements such that

$$e_1 f_1 + e_2 f_1' = z_1 \in Z_1 \tag{1}$$
$$e_1 f_2 + e_2 f_2' = z_2 \in Z_2 \tag{2}$$
$$e_1 f_3 + e_2 f_3' = z_3 \in Z_3 \tag{3}$$

Recovering elements of E

Suppose $\begin{pmatrix} f_1 & f_1' \\ f_2 & f_2' \end{pmatrix}$ invertible, we can recover e_1 and e_2 with

$$e_1 = \frac{\begin{vmatrix} z_1 & f_1' \\ z_2 & f_2' \end{vmatrix}}{\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}} \in \frac{\begin{vmatrix} Z_1 & f_1' \\ Z_2 & f_2' \end{vmatrix}}{\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}} = \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}^{-1} (f_2' Z_1 + f_1' Z_2)$$

Similarly,

$$e_2 \in \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}^{-1} (f_2 Z_1 + f_1 Z_2)$$

Combining signatures two by two

Compute

$$A := \frac{f_2' Z_1 + f_1' Z_2}{\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}} \bigcap \frac{f_3' Z_1 + f_1' Z_3}{\begin{vmatrix} f_1 & f_1' \\ f_3 & f_3' \end{vmatrix}} \bigcap \frac{f_3' Z_2 + f_2' Z_3}{\begin{vmatrix} f_2 & f_2' \\ f_3 & f_3' \end{vmatrix}}$$

With great probability,

- If we are in the case of equations (1), (2) and (3) then $A = Vect(e_1)$
- Else, $A = \{0\}$ and we retry with other random (f_2, f_2', f_3, f_3').

$$\text{Probability of success } \approx 0.25 q^{-4d}$$

Signing process in Durandal

To produce a Durandal signature, we need to solve a system :

$$\boldsymbol{z} = \boldsymbol{c}\boldsymbol{S}' + \boldsymbol{p}\boldsymbol{S}$$

with

- $\boldsymbol{p} \in F^{4k}$ unknown
- $\mathrm{Supp}(\boldsymbol{z}) \subset U$ filtered subspace in EF of codimension λ
- $\boldsymbol{c}$ depending on the message
- $\boldsymbol{S}$ and $\boldsymbol{S}'$ the secret key

Signing process in Durandal

It is shown to be equivalent to solving :

$$\boldsymbol{M} \begin{pmatrix} p_{11} \\ \vdots \\ p_{i\ell} \\ \vdots \\ p_{lkd} \end{pmatrix} = \boldsymbol{b} \tag{4}$$

where $\boldsymbol{M}$ is the binary matrix

$$\boldsymbol{M} = (\pi_h(f_\ell \boldsymbol{S}_{ij}))_{11 \leq i\ell \leq lkd, 11 \leq hj \leq \lambda n} \tag{5}$$

(π_h is the projector on the last λ coordinates of EF)

Naive inversion

$\boldsymbol{M}$ is a large $\lambda n \times \lambda n$ binary matrix.

$$\text{Cost}: O((\lambda n)^{\omega})$$

Spotting structure in $\boldsymbol{M}$

$\boldsymbol{M}$ is composed of ideal blocks $\boldsymbol{M}_{\ell,h} = \pi_h(f_\ell \boldsymbol{S})$

$$\begin{pmatrix} \boldsymbol{M}_{1,1} & \cdots & \boldsymbol{M}_{1,\lambda} \\ & \cdots & \\ & \cdots & \\ \vdots & \boldsymbol{M}_{\ell,h} & \vdots \\ & \cdots & \\ & \cdots & \\ \boldsymbol{M}_{d,1} & & \boldsymbol{M}_{d,\lambda} \end{pmatrix}$$

Spotting structure in M

Each block is of size $k \times k$ and can be inverted with Euclid's algorithm (with cost $O(k \log k)$).

We then use Strassen algorithm :

	Naive	Ours
Cost	$O((\lambda n)^\omega)$	$O(\lambda^\omega n \log n)$

Keygen	Signature	Verification
5ms	~~350ms~~ 40ms	5ms

Variant scheme

Sign

$\boldsymbol{y} \xleftarrow{\$} (W + EF)^n$
$\boldsymbol{x} = \boldsymbol{y}\boldsymbol{H}^\top$

Verify

$\boldsymbol{x} = \boldsymbol{H}\boldsymbol{z}^\top + \boldsymbol{S}'\boldsymbol{c}^\top + \boldsymbol{S}\boldsymbol{p}^\top$

Variant scheme

Sign

$\boldsymbol{y} \xleftarrow{\$} (W + EF)^n$
$\boldsymbol{x} = \boldsymbol{y}\boldsymbol{H}^\top$

Verify

$\boldsymbol{x} = \boldsymbol{H}\boldsymbol{z}^\top + \boldsymbol{S}'\boldsymbol{c}^\top + \boldsymbol{S}\boldsymbol{p}^\top$

Sign

$\hat{\boldsymbol{x}} \xleftarrow{\$} \mathbb{F}_{q^m}^b$
Solve $\hat{\boldsymbol{x}} = \boldsymbol{y}\hat{\boldsymbol{H}}^\top$ with
$\mathrm{Supp}(\boldsymbol{y}) = W + EF$
$\boldsymbol{x} = \boldsymbol{y}\boldsymbol{H}^\top$

Verify

Solve
$\hat{\boldsymbol{x}} = \hat{\boldsymbol{H}}\boldsymbol{z}^\top + \hat{\boldsymbol{S}}'\boldsymbol{c}^\top + \hat{\boldsymbol{S}}\boldsymbol{p}^\top$ with
$\mathrm{Supp}(\boldsymbol{z})$
$\boldsymbol{x} = \boldsymbol{H}\boldsymbol{z}^\top + \boldsymbol{S}'\boldsymbol{c}^\top + \boldsymbol{S}\boldsymbol{p}^\top$