Analysis of the security of the PSSI problem and applications and optimizations to Durandal signature scheme

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Durandal signature scheme

- Rank-based signature presented at EUROCRYPT'19 [ABG$^+$19]
- Adaptation of Schnorr-Lyubashevsky proof of knowledge, with variations to avoid attacks
- Fiat-Shamir heuristic to transform into a signature scheme
- No equivalent found for Hamming metric
- Based on problems : RSL, IRSD, **PSSI**

	pk size	σ size
Durandal-I	15.2KB	4.1KB
Durandal-II	18.6KB	5.0KB

What has happened with Durandal since 2019 ?

- Resistant to attacks since 2019
- Better understanding of the RSL problem (algebraic attack in 2021 [BB21], combinatorial attack in 2022 [BBBG22])
- PSSI reduction to MinRank (ongoing work)
- New combinatorial attack on PSSI (ongoing work, breaks existing parameters in $\approx 2^{36}$ attempts)
- Optimizations and size-performance tradeoffs

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Summary

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Notation

- $\mathbf{Gr}(d, \mathbb{F}_{q^m})$ is the set of subspaces of $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$-dimension d.
- $x \overset{\$}{\leftarrow} X$ means that x is chosen uniformly at random in X
- For E, F subspaces of $\mathbb{F}_{q^m}$, the product space EF is defined as :

$$EF := Vect_{\mathbb{F}_q}\{ef | e \in E, f \in F\}$$

If $(e_1, ..., e_r)$ and $(f_1, ..., f_d)$ are basis of E and F, then $(e_i f_j)_{1 \leq i \leq r, 1 \leq j \leq d}$ contains a basis of EF.

PSSI problem

Definition (PSS sample)

Let $E \subset \mathbb{F}_{q^m}$ a subspace of $\mathbb{F}_q$-dimension r. A Product Space Subspace (PSS) sample is a couple of subspaces (F, Z) defined as follows :

- $F \xleftarrow{\$} \mathbf{Gr}(d, \mathbb{F}_{q^m})$
- $U \xleftarrow{\$} \mathbf{Gr}(rd - \lambda, EF)$ such that $\{ef | e \in E, f \in F\} \cap U = \{0\}$
- $W \xleftarrow{\$} \mathbf{Gr}(w, \mathbb{F}_{q^m})$
- $Z = W + U$

PSSI problem

Definition (Random sample)

A random sample is a couple of subspaces (F, Z) with :

- $F \xleftarrow{\$} \mathbf{Gr}(d, \mathbb{F}_{q^m})$
- $Z \xleftarrow{\$} \mathbf{Gr}(w + rd - \lambda, \mathbb{F}_{q^m})$
- F and Z are independent

PSSI problem

Definition (PSSI problem, from Durandal [ABG+19])

The Product Spaces Subspaces Indistinguishability (PSSI) problem consists in deciding whether N samples (F_i, Z_i) are PSS samples or random samples.

Definition (Search-PSSI problem)

Given N PSS samples (F_i, Z_i), the search-PSSI problem consists in finding the vector space E of dimension r.

What happens if $\lambda = 0$?

There is no filtration : $(F, Z) = (F, W + EF)$.
Take $(f_1, \ldots, f_d)$ a basis of F.

To find E in one sample, compute :

$$A = \bigcap_{i=1}^{d} f_i^{-1} Z$$

Similar arguments than LRPC decoding :

$$\begin{aligned} f_i^{-1} Z &= f_i^{-1} f_1 E + \ldots + E + \ldots + f_i^{-1} f_d E + f_i^{-1} W \\ &= E + R_i \end{aligned}$$

Practical parameters for PSSI

m	w	r	d	λ
241	57	6	6	12

Existing attack for PSSI

Choose $A \subset F$ a subspace of dimension 2 and check whether

$$\dim(AZ) < 2(w + rd - \lambda)$$

Proposition ([ABG+19])

The advantage of the distinguisher is of the order of $q^{(rd-\lambda)-m}$.

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Proposition ([ABG+19])

The advantage of the distinguisher is of the order of $q^{(rd-\lambda)-m}$.

Several problems :

- The distinguisher only uses **one** signature ;
- It does not depend on w ;
- It does not allow to recover the secret space E.

Summary

Combining two instances

A partial explanation

Impossibility to avoid 2-sums (for $\lambda = 2r = 2d$)

Protection by m

Recall that

- $\dim F = d$
- $\dim Z = w + rd - \lambda$

so

$$\dim F_1 Z_2 + F_2 Z_1 = 2d(w + rd - \lambda) > m$$

but we can take subspaces of F_1 and F_2 to remain below m!

m	w	r	d	λ	$w + rd - \lambda$
241	57	6	6	12	81

Summary

Refining the first observation

Probability

Heuristic

Let $(e_1, e_2) \xleftarrow{\$} E$ and $U \subset EF$ filtered of dimension $rd - \lambda$.
Suppose $\lambda = 2d$, then

$$\mathbb{P}(\exists (f_1, f_2) \in F \mid e_1 f_1 + e_2 f_2 \in U) \geq 1 - \frac{1}{e}$$

Heuristic

Let $(f_1, f_2) \xleftarrow{\$} F$ and $U \subset EF$ filtered of dimension $rd - \lambda$.
Suppose $\lambda = 2r$, then

$$\mathbb{P}(\exists (e_1, e_2) \in E \mid e_1 f_1 + e_2 f_2 \in U) \geq 1 - \frac{1}{e}$$

An attack with three signatures

Recovering elements of E

Combining signatures two by two

Does it really work ?

We want the chain of intersections

$$A := \frac{f_2' Z_1 + f_1' Z_2}{\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}} \bigcap \frac{f_3' Z_1 + f_1' Z_3}{\begin{vmatrix} f_1 & f_1' \\ f_3 & f_3' \end{vmatrix}} \bigcap \frac{f_3' Z_2 + f_2' Z_3}{\begin{vmatrix} f_2 & f_2' \\ f_3 & f_3' \end{vmatrix}}$$

to be equal to $\{0\}$, in general.

All the subspaces $f_i Z_j + f_j Z_i$ are of dimension $2(w + rd - \lambda)$.

m	w	r	d	λ	$2(w + rd - \lambda)$
241	57	6	6	12	162

Probabilities on the intersection of two vector spaces

Heuristic

Let A and B be uniformly random and independent subspaces of $\mathbb{F}_{q^m}$ of dimension a and b, respectively.

- If $a + b < m$, then $\mathbb{P}(\dim(A \cap B) > 0) \approx q^{a+b-m}$;
- If $a + b \geq m$, then the most probable outcome is $\dim(A \cap B) = a + b - m$.

Generalization to n intersections

Heuristic

For $1 \leq i \leq n$, let $A_i \xleftarrow{\$} Fqm$ be independent subspaces of fixed dimension a.

- If $na < (n-1)m$, then $\mathbb{P}(\dim(\bigcap_{i=1}^n A_i) > 0) \approx q^{na-(n-1)m}$;
- If $na \geq (n-1)m$, then the most probable outcome is $\dim(\bigcap_{i=1}^n A_i) = na - (n-1)m$;

In our setting :

- $a = 162, m = 241, n = 3$
- $na = 486, (n-1)m = 482$

Most probable outcome : $\dim(A) = 4$

☹

Let's refine again !

We consider four samples :

$$(F_1, Z_1), (F_2, Z_2), (F_3, Z_3), (F_4, Z_4)$$

and we draw matrices :

$$\begin{pmatrix} f_1 & f_1' \\ f_2 & f_2' \\ f_3 & f_3' \\ f_4 & f_4' \end{pmatrix}$$

with (f_1, f_1') fixed.

Probability of success $\approx (1 - 1/e)^4 q^{-6d} \approx 0.16 q^{-6d}$

And now 6 vectors spaces to intersect !

Summary

Probability of success of the attack

$$\approx 0.16q^{-6d}$$

Increase λ	$\Rightarrow$	Impossible due to inexistence of solution
Decrease m	$\Rightarrow$	Impossible due to Singleton bound
Increase d	$\Rightarrow$	Very large parameters... ($m \geq 400$)

Increase q !

New parameters

q	m	k	n	w	r	d	λ
2	241	101	202	57	6	6	12

pk size	σ size	MaxMinors [BBC+20]	Our attack
15.2KB	4.1KB	98	56

$$\downarrow$$

q	m	k	n	w	r	d	λ
4	173	85	170	5	8	9	18

pk size	σ size	MaxMinors [BBC+20]	Our attack
14.7KB	5.1KB	232	128

Keygen	Signature	Verification
5ms	350ms	2ms

1

Summary

Two optimizations

1. Fast matrix inversion for signing
2. Size-performance tradeoff

Spotting structure in a linear system

Idea

Exploit a block structure in a big linear system of size $\lambda n \times \lambda n$.

Each block is of size $k \times k$ and can be inverted with Euclid's algorithm (with cost $O(k \log k)$).

We then use Strassen algorithm :

	Naive	Ours
Cost	$O((\lambda n)^\omega)$	$O(\lambda^\omega n \log n)$

Keygen	Signature	Verification
5ms	~~350ms~~ **40ms**	2ms

Size-performance tradeoff

A Durandal signature is composed of a tuple $(\boldsymbol{z}, \boldsymbol{c}, \boldsymbol{p})$. To verify, compute

$$\boldsymbol{x} = \boldsymbol{H}\boldsymbol{z}^\top + \boldsymbol{S}'\boldsymbol{c}^\top + \boldsymbol{S}\boldsymbol{p}^\top$$

Idea

Send $\mathrm{Supp}(\boldsymbol{z})$ instead of $\boldsymbol{z}$.

...but we also need to send some coordinates of $\boldsymbol{x}$!

Impact on parameters

q	m	k	n	w	r	d	λ
4	173	85	170	5	8	9	18

Shorter size version

pk size	σ size	Signing time	Verification time
14.7KB	~~5.1KB~~ **2.6KB**	~~40ms~~ **1s**	~~2ms~~ **1s**

Faster verification version

pk size	σ size	Signing time	Verification time
14.7KB	~~5.1KB~~ **4.3KB**	~~40ms~~ **1s**	2ms

Summary

Conclusion

- Analysis of a less studied problem at the core of a competitive signature scheme
- New secure parameters remain attractive
- Optimizations makes the scheme even more competitive

Perspectives

- Refine the analysis on the security of PSSI problem
- Tweak to avoid the new attack on PSSI without penalizing the parameters

Thank you for your attention !

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Combining two instances

We simplify and assume $w = 0$.
We take two instances $(F_1, Z_1), (F_2, Z_2)$.
We made the following observation :

- Z_1 is filtered in EF_1
- Z_2 is filtered in EF_2
- but...
- $F_1 Z_2 + F_2 Z_1$ is not filtered in $E(F_1 F_2)$!

A partial explanation

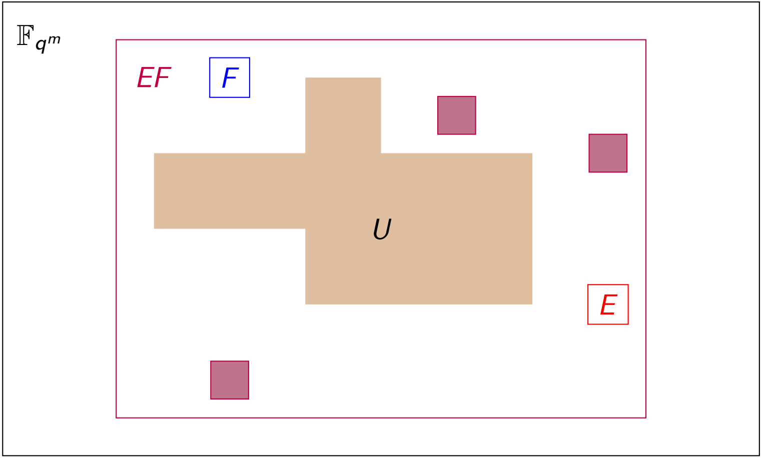
If there exists $(e_1, e_2) \in E^2$ such that

$$e_1 f_1 + e_2 f_1' = z_1 \in Z_1$$
$$e_1 f_2 + e_2 f_2' = z_2 \in Z_2$$

then

$$f_1' z_2 + f_2' z_1 = e_1(f_1' f_2 + f_2' f_1)$$

Impossibility to avoid 2-sums



Refining the first observation

By drawing randomly a matrix

$$\begin{pmatrix} f_1 & f_1' \\ f_2 & f_2' \end{pmatrix} \quad (f_1, f_1') \xleftarrow{\$} F_1, (f_2, f_2') \xleftarrow{\$} F_2$$

we get (roughly) q^{-4d} chances of having a product element ef (with $e \in E, f \in F_1F_2$) :

$$ef \in f_1' Z_2 + f_2' Z_1$$

Refining the first observation

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$$ef \in f_1' Z_2 + f_2' Z_1$$

We need :

- A way to recover this element $e \in E$;
- A precise probability of recovering e

The attack

We consider three samples :

$$(F_1, Z_1)$$
$$(F_2, Z_2)$$
$$(F_3, Z_3)$$

Let $(f_1, f_1') \xleftarrow{\$} F_1$. With probability greater than

$$(1 - 1/e)^3 \approx 0,25$$

there exists elements such that

$$e_1 f_1 + e_2 f_1' = z_1 \in Z_1 \quad (1)$$
$$e_1 f_2 + e_2 f_2' = z_2 \in Z_2 \quad (2)$$
$$e_1 f_3 + e_2 f_3' = z_3 \in Z_3 \quad (3)$$

Recovering elements of E

Suppose $\begin{pmatrix} f_1 & f_1' \\ f_2 & f_2' \end{pmatrix}$ invertible, we can recover e_1 and e_2 with

$$e_1 = \frac{\begin{vmatrix} z_1 & f_1' \\ z_2 & f_2' \end{vmatrix}}{\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}} \in \frac{\begin{vmatrix} Z_1 & f_1' \\ Z_2 & f_2' \end{vmatrix}}{\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}} = \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}^{-1} (f_2' Z_1 + f_1' Z_2)$$

Similarly,

$$e_2 \in \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}^{-1} (f_2 Z_1 + f_1 Z_2)$$

Combining signatures two by two

Compute

$$A := \frac{f_2' Z_1 + f_1' Z_2}{\begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}} \bigcap \frac{f_3' Z_1 + f_1' Z_3}{\begin{vmatrix} f_1 & f_1' \\ f_3 & f_3' \end{vmatrix}} \bigcap \frac{f_3' Z_2 + f_2' Z_3}{\begin{vmatrix} f_2 & f_2' \\ f_3 & f_3' \end{vmatrix}}$$

With great probability,

- If we are in the case of equations (1), (2) and (3) then $A = Vect(e_1)$
- Else, $A = \{0\}$ and we retry with other random (f_2, f_2', f_3, f_3').

$$\text{Probability of success } \approx 0.25q^{-4d}$$

Signing process in Durandal

To produce a Durandal signature, we need to solve a system :

$$\boldsymbol{z} = \boldsymbol{c}\boldsymbol{S}' + \boldsymbol{p}\boldsymbol{S}$$

with

- $\boldsymbol{p} \in F^{4k}$ unknown
- $\mathrm{Supp}(\boldsymbol{z}) \subset U$ filtered subspace in EF of codimension λ
- $\boldsymbol{c}$ depending on the message
- $\boldsymbol{S}$ and $\boldsymbol{S}'$ the secret key

Signing process in Durandal

It is shown to be equivalent to solving :

$$\boldsymbol{M} \begin{pmatrix} p_{11} \\ \vdots \\ p_{i\ell} \\ \vdots \\ p_{lkd} \end{pmatrix} = \boldsymbol{b} \tag{4}$$

where $\boldsymbol{M}$ is the binary matrix

$$\boldsymbol{M} = (\pi_h(f_\ell \boldsymbol{S}_{ij}))_{11 \leq i\ell \leq lkd, 11 \leq hj \leq \lambda n} \tag{5}$$

(π_h is the projector on the last λ coordinates of EF)

Naive inversion

$\boldsymbol{M}$ is a large $\lambda n \times \lambda n$ binary matrix.

$$\text{Cost} : O((\lambda n)^{\omega})$$

Spotting structure in $\boldsymbol{M}$

$\boldsymbol{M}$ is composed of ideal blocks $\boldsymbol{M}_{\ell,h} = \pi_h(f_\ell \boldsymbol{S})$

$$
\left(\begin{array}{c|c|c}
\boldsymbol{M}_{1,1} & \cdots & \boldsymbol{M}_{1,\lambda} \\
\hline
 & \cdots & \\
\hline
\vdots & \boldsymbol{M}_{\ell,h} & \vdots \\
\hline
 & \cdots & \\
\hline
\boldsymbol{M}_{d,1} & \cdots & \boldsymbol{M}_{d,\lambda}
\end{array}\right)
$$

Spotting structure in M

Each block is of size $k \times k$ and can be inverted with Euclid's algorithm (with cost $O(k \log k)$).

We then use Strassen algorithm :

	Naive	Ours
Cost	$O((\lambda n)^\omega)$	$O(\lambda^\omega n \log n)$

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Variant scheme

Sign

$\boldsymbol{y} \xleftarrow{\$} (W + EF)^n$
$\boldsymbol{x} = \boldsymbol{y}\boldsymbol{H}^\top$

Verify

$$\boldsymbol{x} = \boldsymbol{H}\boldsymbol{z}^\top + \boldsymbol{S}'\boldsymbol{c}^\top + \boldsymbol{S}\boldsymbol{p}^\top$$

Variant scheme

Sign

$\boldsymbol{y} \xleftarrow{\$} (W + EF)^n$
$\boldsymbol{x} = \boldsymbol{y}\boldsymbol{H}^\top$

Verify

$\boldsymbol{x} = \boldsymbol{H}\boldsymbol{z}^\top + \boldsymbol{S}'\boldsymbol{c}^\top + \boldsymbol{S}\boldsymbol{p}^\top$

Sign

$\hat{\boldsymbol{x}} \xleftarrow{\$} \mathbb{F}_{q^m}^b$
Solve $\hat{\boldsymbol{x}} = \boldsymbol{y}\hat{\boldsymbol{H}}^\top$ with
$\mathrm{Supp}(\boldsymbol{y}) = W + EF$
$\boldsymbol{x} = \boldsymbol{y}\boldsymbol{H}^\top$

Verify

Solve
$\hat{\boldsymbol{x}} = \hat{\boldsymbol{H}}\boldsymbol{z}^\top + \hat{\boldsymbol{S}}'\boldsymbol{c}^\top + \hat{\boldsymbol{S}}\boldsymbol{p}^\top$ with
$\mathrm{Supp}(\boldsymbol{z})$
$\boldsymbol{x} = \boldsymbol{H}\boldsymbol{z}^\top + \boldsymbol{S}'\boldsymbol{c}^\top + \boldsymbol{S}\boldsymbol{p}^\top$